Math 232 Name: Key Dr. Beezer
Quiz SLE Code: Fall 2004

Show *all* of your work and *explain* your answers fully. There is a total of 85 possible points.

1. Row-reduce the matrix below without the aid of a calculator, indicating the row operations you are using at each step. (15 points)

$$\begin{bmatrix} 2 & 1 & 5 & 10 \\ 1 & -3 & -1 & -2 \\ 4 & -2 & 6 & 12 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} 2 & 1 & 5 & 10 \\ 1 & -3 & -1 & -2 \\ 4 & -2 & 6 & 12 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & -3 & -1 & -2 \\ 2 & 1 & 5 & 10 \\ 4 & -2 & 6 & 12 \end{bmatrix}$$

$$\xrightarrow{-2R_1+R_2} \begin{bmatrix} 1 & -3 & -1 & -2 \\ 0 & 7 & 7 & 14 \\ 4 & -2 & 6 & 12 \end{bmatrix} \xrightarrow{-4R_1+R_3} \begin{bmatrix} 1 & -3 & -1 & -2 \\ 0 & 7 & 7 & 14 \\ 0 & 10 & 10 & 20 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{7}R_2} \begin{bmatrix} 1 & -3 & -1 & -2 \\ 0 & 1 & 1 & 2 \\ 0 & 10 & 10 & 20 \end{bmatrix} \xrightarrow{3R_2+R_1} \begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 10 & 10 & 20 \end{bmatrix}$$

$$\xrightarrow{-10R_2+R_3} \begin{bmatrix} \boxed{1} & 0 & 2 & 4 \\ 0 & \boxed{1} & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

2. Is the matrix below singular or nonsingular? Why? (10 points)

$$\begin{bmatrix} -3 & 1 & 2 & 8 \\ 2 & 0 & 3 & 4 \\ 1 & 2 & 7 & -4 \\ 5 & -1 & 2 & 0 \end{bmatrix}$$

Solution: The matrix row-reduces to

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

which is the 4×4 identity matrix. By Theorem NSRRI the original matrix must be nonsingular.

3. Find all solutions to each of the following systems of linear equations. Write each solution set as a set, using the correct set notation. You may use your calculator to row-reduce any matrices, being certain to write down the input matrix, the command used, and the output. (30 points)
(a)

$$\begin{aligned} 2x_1 - 3x_2 + x_3 + 7x_4 &= 14 \\ 2x_1 + 8x_2 - 4x_3 + 5x_4 &= -1 \\ x_1 + 3x_2 - 3x_3 &= 4 \\ -5x_1 + 2x_2 + 3x_3 + 4x_4 &= -19 \end{aligned}$$

Solution: The augmented matrix row-reduces to

$$\begin{bmatrix} \boxed{1} & 0 & 0 & 0 & 1 \\ 0 & \boxed{1} & 0 & 0 & -3 \\ 0 & 0 & \boxed{1} & 0 & -4 \\ 0 & 0 & 0 & \boxed{1} & 1 \end{bmatrix}$$

and we see from the locations of the leading 1's that the system is consistent (Theorem RCLS) and that $n - r = 4 - 4 = 0$ and so the system has no free variables (Theorem CSRN) and hence has a unique solution. This solution is $\{(1, -3, -4, 1)\}$.

(b)

$$\begin{aligned} 3x_1 + 4x_2 - x_3 + 2x_4 &= 6 \\ x_1 - 2x_2 + 3x_3 + x_4 &= 2 \\ 10x_2 - 10x_3 - x_4 &= 1 \end{aligned}$$

Solution: The augmented matrix row-reduces to

$$\begin{bmatrix} \boxed{1} & 0 & 1 & 4/5 & 0 \\ 0 & \boxed{1} & -1 & -1/10 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \end{bmatrix}$$

and a leading 1 in the last column tells us that the system is inconsistent (Theorem RCLS). So the solution set is $\emptyset = \{\}$.

(c)

$$\begin{aligned} 2x_1 + 4x_2 + 5x_3 + 7x_4 &= -26 \\ x_1 + 2x_2 + x_3 - x_4 &= -4 \\ -2x_1 - 4x_2 + x_3 + 11x_4 &= -10 \end{aligned}$$

Solution: The augmented matrix row-reduces to

$$\begin{bmatrix} \boxed{1} & 2 & 0 & -4 & 2 \\ 0 & 0 & \boxed{1} & 3 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(Theorem RCLS) and (Theorem CSRN) tells us the system is consistent and the solution set can be described with $n - r = 4 - 2 = 2$ free variables, namely x_2 and x_4. Solving for the dependent variables ($D = \{x_1, x_3\}$) the first and second equations represented in the row-reduced matrix yields,

$$\begin{aligned} x_1 &= 2 - 2x_2 + 4x_4 \\ x_3 &= -6 \qquad - 3x_4 \end{aligned}$$

As a set, we write this as

$$\{(2 - 2x_2 + 4x_4,\ x_2,\ -6 - 3x_4,\ x_4) \mid x_2,\ x_4 \in \mathbb{C}\}$$

4. For each system of linear equations described below, say **as much as possible** about its solution set. Be sure to make it clear which theorems you are using to reach your conclusions (you do not need to quote their acronyms, though). (15 points)
(a) A homogeneous system of 8 equations in 8 variables.

Solution: Since the system is homogeneous, we know it has the trivial solution (Theorem HSC). We cannot say anymore based on the information provided, except to say that there is either a unique solution or infinitely many solutions (Theorem PSSLS). See Archetype A and Archetype B to understand the possibilities.

(b) A consistent system of 8 equations in 6 variables.

Solution: Consistent means there is at least one solution (Definition CS). It will have either a unique solution or infinitely many solutions (Theorem PSSLS).

(c) A consistent system of 6 equations in 8 variables.

Solution: With 6 rows in the augmented matrix, the row-reduced version will have $r \leq 6$. Since the system is consistent, apply Theorem CSRN to see that $n - r \geq 2$ implies infinitely many solutions.

5. The Thompson Hall parking lot has 66 vehicles (cars, trucks, motorcycles and bicycles) in it at the moment. There are four times as many cars as trucks. The total number of tires (4 per car or truck, 2 per motorcycle or bicycle) is 252. How many cars are there? How many bicycles? (15 points)

Solution: Let c, t, m, b denote the number of cars, trucks, motorcycles, and bicycles. Then the statements from the problem yield the equations:

$$\begin{aligned} c + t + m + b &= 66 \\ c - 4t &= 0 \\ 4c + 4t + 2m + 2b &= 252 \end{aligned}$$

The augmented matrix for this system is

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 66 \\ 1 & -4 & 0 & 0 & 0 \\ 4 & 4 & 2 & 2 & 252 \end{bmatrix}$$

which row-reduces to

$$\begin{bmatrix} \boxed{1} & 0 & 0 & 0 & 48 \\ 0 & \boxed{1} & 0 & 0 & 12 \\ 0 & 0 & \boxed{1} & 1 & 6 \end{bmatrix}$$

$c = 48$ is the first equation represented in the row-reduced matrix so there are 48 cars. $m + b = 6$ is the third equation represented in the row-reduced matrix so there are anywhere from 0 to 6 bicycles. We can also say that b is a free variable, but the context of the problem limits it to 7 integer values since cannot have a negative number of motorcycles.