Math 232 Name: Key Dr. Beezer
Quiz V Fall 2004

Show *all* of your work and *explain* your answers fully. There is a total of 95 possible points.

1. Find the vector form of the solutions to the system of equations below. (15 points)

$$\begin{aligned} 2x_1 - 4x_2 + 3x_3 + x_5 &= 6 \\ x_1 - 2x_2 - 2x_3 + 14x_4 - 4x_5 &= 15 \\ x_1 - 2x_2 + x_3 + 2x_4 + x_5 &= -1 \\ -2x_1 + 4x_2 - 12x_4 + x_5 &= -7 \end{aligned}$$

Solution: Row-reduce the augmented matrix representing this system, to find

$$\begin{bmatrix} \boxed{1} & -2 & 0 & 6 & 0 & 1 \\ 0 & 0 & \boxed{1} & -4 & 0 & 3 \\ 0 & 0 & 0 & 0 & \boxed{1} & -5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The system is consistent (no leading one in column 6, Theorem RCLS). x_2 and x_4 are the free variables. Now apply Theorem VFSLS, or follow the three-step process of Example SS.VFSADI, Example SS.VFSAL to obtain

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 0 \\ -5 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -6 \\ 0 \\ 4 \\ 1 \\ 0 \end{bmatrix}$$

2. For the matrix A below, find a set of vectors S so that (1) S is linearly independent, and (2) the span of S equals the null space of A, $\mathrm{Sp}(S) = \mathrm{N}(A)$. (15 points)

$$A = \begin{bmatrix} 1 & 1 & 6 & -8 \\ 1 & -2 & 0 & 1 \\ -2 & 1 & -6 & 7 \end{bmatrix}$$

Solution: Theorem BNS says that if we find the vector form of the solutions to the homogeneous system $\mathcal{LS}(A, \mathbf{0})$, then the fixed vectors (one per free variable) will have the desired properties. Row-reduce A, viewing it as the augmented matrix of a homogeneous system with an invisible columns of zeros as the last column,

$$\begin{bmatrix} \boxed{1} & 0 & 4 & -5 \\ 0 & \boxed{1} & 2 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Moving to the vector form of the solutions (Theorem VFSLS), with free variables x_3 and x_4, solutions to the consistent system (it is homogenous, Theorem HSC) can be expressed as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -4 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 5 \\ 3 \\ 0 \\ 1 \end{bmatrix}$$

Then with S given by

$$S = \left\{ \begin{bmatrix} -4 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Theorem BNS guarantees the set has the desired properties.

3. Suppose that $S = \left\{ \begin{bmatrix} 2 \\ -1 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ -2 \\ 1 \end{bmatrix} \right\}$. Let $W = \mathrm{Sp}\,(S)$. (15 points)

(a) Let $\mathbf{x} = \begin{bmatrix} 5 \\ 8 \\ -12 \\ -5 \end{bmatrix}$. Is $\mathbf{x} \in W$? If so, provide an explicit linear combination that demonstrates this.

Solution: Rephrasing the question, we want to know if there are scalars α_1 and α_2 such that

$$\alpha_1 \begin{bmatrix} 2 \\ -1 \\ 3 \\ 4 \end{bmatrix} + \alpha_2 \begin{bmatrix} 3 \\ 2 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \\ -12 \\ -5 \end{bmatrix}$$

Theorem SLSLC allows us to rephrase the question again as a quest for solutions to the system of four equations in two unknowns with an augmented matrix given by

$$\begin{bmatrix} 2 & 3 & 5 \\ -1 & 2 & 8 \\ 3 & -2 & -12 \\ 4 & 1 & -5 \end{bmatrix}$$

This matrix row-reduces to

$$\begin{bmatrix} \boxed{1} & 0 & -2 \\ 0 & \boxed{1} & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

From the form of this matrix, we can see that $\alpha_1 = -2$ and $\alpha_2 = 3$ is an affirmative answer to our question. More convincingly,

$$(-2) \begin{bmatrix} 2 \\ -1 \\ 3 \\ 4 \end{bmatrix} + (3) \begin{bmatrix} 3 \\ 2 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \\ -12 \\ -5 \end{bmatrix}$$

(b) Let $\mathbf{y} = \begin{bmatrix} 5 \\ 1 \\ 3 \\ 5 \end{bmatrix}$. Is $\mathbf{y} \in W$? If so, provide an explicit linear combination that demonstrates this.

Solution: Rephrasing the question, we want to know if there are scalars α_1 and α_2 such that

$$\alpha_1 \begin{bmatrix} 2 \\ -1 \\ 3 \\ 4 \end{bmatrix} + \alpha_2 \begin{bmatrix} 3 \\ 2 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 3 \\ 5 \end{bmatrix}$$

Theorem SLSLC allows us to rephrase the question again as a quest for solutions to the system of four equations in two unknowns with an augmented matrix given by

$$\begin{bmatrix} 2 & 3 & 5 \\ -1 & 2 & 1 \\ 3 & -2 & 3 \\ 4 & 1 & 5 \end{bmatrix}$$

This matrix row-reduces to

$$\begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \end{bmatrix}$$

With a leading 1 in the last column of this matrix (Theorem RCLS) we can see that the system of equations has no solution, so there are no values for α_1 and α_2 that will allow us to conclude that $\mathbf{y}$ is in W. So $\mathbf{y} \notin W$.

4. Determine if the sets of vectors below are linearly independent or linearly dependent. (20 points)

(a) $\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 0 \end{bmatrix} \right\}$

Solution: With three vectors from $\mathbb{C}^3$, we can form a square matrix by making these three vectors the columns of a matrix. We do so, and row-reduce to obtain,

$$\begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \end{bmatrix}$$

the 3×3 identity matrix. So by NSME2 the original matrix is nonsingular and its columns are therefore a linearly independent set.

(b) $\left\{ \begin{bmatrix} -1 \\ 2 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 7 \\ 3 \\ -6 \\ 4 \end{bmatrix} \right\}$

Solution: Corollary LIVRN says we can answer this question by putting theses vectors into a matrix as columns and row-reducing. Doing this we obtain,

$$\begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \end{bmatrix}$$

With $n = 3$ (3 vectors, 3 columns) and $r = 3$ (3 leading 1's) we have $n = r$ and the corollary says the vectors are linearly independent.

(c) $\left\{ \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 9 \\ -3 \\ 8 \end{bmatrix}, \begin{bmatrix} 2 \\ 8 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix} \right\}$

Solution: Five vectors from $\mathbb{C}^3$. Theorem MVSLD says the set is linearly dependent. Boom.

5. Consider the set of vectors from $\mathbb{C}^3$, W, given below. Find a set T that contains three vectors and $W = \mathrm{Sp}\,(T)$. (15 points)

$$W = \mathrm{Sp}\left(\{\mathbf{v}_1,\, \mathbf{v}_2,\, \mathbf{v}_3,\, \mathbf{v}_4,\, \mathbf{v}_5\}\right) = \mathrm{Sp}\left(\left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} \right\} \right)$$

Solution: We want to find some relations of linear dependence on $\{\mathbf{v}_1,\, \mathbf{v}_2,\, \mathbf{v}_3,\, \mathbf{v}_4,\, \mathbf{v}_5\}$ that will allow us to "kick out" some vectors, in the spirit of Example SS.SCAD and Example LI.RS. To find relations of linear dependence, we formulate a matrix A whose columns are $\mathbf{v}_1,\, \mathbf{v}_2,\, \mathbf{v}_3,\, \mathbf{v}_4,\, \mathbf{v}_5$. Then we consider the homogeneous sytem of equations $\mathcal{LS}(A,\, \mathbf{0})$ by row-reducing its coefficient matrix (remember that if we formulated the augmented matrix we would just add a column of zeros). After row-reducing, we obtain

$$\begin{bmatrix} \boxed{1} & 0 & 0 & 2 & -1 \\ 0 & \boxed{1} & 0 & 1 & -2 \\ 0 & 0 & \boxed{1} & 0 & 0 \end{bmatrix}$$

From this we that solutions can be obtained employing the free variables x_4 and x_5. With appropriate choices we will be able to conclude that vectors $\mathbf{v}_4$ and $\mathbf{v}_5$ are unnecessary for creating W via a span. By Theorem SLSLC the choice of free variables below lead to solutions and linear combinations, which are then rearranged.

$$\begin{aligned} x_4 = 1, x_5 = 0 \quad &\Rightarrow \quad (-2)\mathbf{v}_1 + (-1)\mathbf{v}_2 + (0)\mathbf{v}_3 + (1)\mathbf{v}_4 + (0)\mathbf{v}_5 = \mathbf{0} \quad \Rightarrow \quad \mathbf{v}_4 = 2\mathbf{v}_1 + \mathbf{v}_2 \\ x_4 = 0, x_5 = 1 \quad &\Rightarrow \quad (1)\mathbf{v}_1 + (2)\mathbf{v}_2 + (0)\mathbf{v}_3 + (0)\mathbf{v}_4 + (1)\mathbf{v}_5 = \mathbf{0} \quad \Rightarrow \quad \mathbf{v}_5 = -\mathbf{v}_1 - 2\mathbf{v}_2 \end{aligned}$$

Since $\mathbf{v}_4$ and $\mathbf{v}_5$ can be expressed as linear combinations of $\mathbf{v}_1$ and $\mathbf{v}_2$ we can say that $\mathbf{v}_4$ and $\mathbf{v}_5$ are not needed for the linear combinations used to build W. Thus

$$W = \mathrm{Sp}\left(\{\mathbf{v}_1,\, \mathbf{v}_2,\, \mathbf{v}_3\}\right) = \mathrm{Sp}\left(\left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\} \right)$$

There are other answers to this question, but notice that any nontrvial linear combination of $\mathbf{v}_1,\, \mathbf{v}_2,\, \mathbf{v}_3,\, \mathbf{v}_4,\, \mathbf{v}_5$ will have a zero coefficient on $\mathbf{v}_3$, so this vector can never be eliminated from the set used to build the span.

Though it was not requested in the problem, notice that $\{\mathbf{v}_1,\, \mathbf{v}_2,\, \mathbf{v}_3\}$ is a linearly independent set, so we cannot make the set any smaller and still span W.

6. Suppose that $\{\mathbf{v}_1,\, \mathbf{v}_2,\, \mathbf{v}_3,\, \mathbf{v}_4\}$ is a linearly independent set in $\mathbb{C}^{35}$. Prove that

$$\{\mathbf{v}_1,\, \mathbf{v}_1 + \mathbf{v}_2,\, \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3,\, \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4\}$$

is a linearly independent set. (15 points)

Solution: Our hypothesis and our conclusion use the term linear independence, so it will get a workout. To establish linear independence, we begin with the definition (Definition LICV) and write a relation of linear dependence (Definition RLDCV),

$$\alpha_1\left(\mathbf{v}_1\right)+\alpha_2\left(\mathbf{v}_1+\mathbf{v}_2\right)+\alpha_3\left(\mathbf{v}_1+\mathbf{v}_2+\mathbf{v}_3\right)+\alpha_4\left(\mathbf{v}_1+\mathbf{v}_2+\mathbf{v}_3+\mathbf{v}_4\right)=\mathbf{0}$$

Using the distributive and commutative properties of vector addition and scalar multiplication (Theorem VSPCM) this equation can be rearranged as

$$\left(\alpha_1+\alpha_2+\alpha_3+\alpha_4\right)\mathbf{v}_1+\left(\alpha_2+\alpha_3+\alpha_4\right)\mathbf{v}_2+\left(\alpha_3+\alpha_4\right)\mathbf{v}_3+\left(\alpha_4\right)\mathbf{v}_4=\mathbf{0}$$

However, this is a relation of linear dependence (Definition RLDCV) on a linearly independent set, $\{\mathbf{v}_1,\,\mathbf{v}_2,\,\mathbf{v}_3,\,\mathbf{v}_4\}$ (this was our lone hypothesis). By the definition of linear independence (Definition LICV) the scalars must all be zero. This is the homogeneous system of equations,

$$\begin{aligned}\alpha_1+\alpha_2+\alpha_3+\alpha_4&=0\\ \alpha_2+\alpha_3+\alpha_4&=0\\ \alpha_3+\alpha_4&=0\\ \alpha_4&=0\end{aligned}$$

Row-reducing the coefficient matrix of this system (or backsolving) gives the conclusion

$$\alpha_1=0 \qquad \alpha_2=0 \qquad \alpha_3=0 \qquad \alpha_4=0$$

This means, by Definition LICV, that the original set

$$\{\mathbf{v}_1,\,\mathbf{v}_1+\mathbf{v}_2,\,\mathbf{v}_1+\mathbf{v}_2+\mathbf{v}_3,\,\mathbf{v}_1+\mathbf{v}_2+\mathbf{v}_3+\mathbf{v}_4\}$$

is linearly independent.