Math 232 Name: Key Dr. Beezer
Quiz M Fall 2004

Show *all* of your work and *explain* your answers fully. There is a total of 90 possible points.

1. Compute the product of the two matrices below, AB. Do this using the definitions of the matrix-vector product (Definition MVP) and the definition of matrix multiplication (Definition MM), no credit will be given for an entry-by-entry computation or a calculator answer. (15 points)

$$A = \begin{bmatrix} 2 & 5 \\ -1 & 3 \\ 2 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 5 & -3 & 4 \\ 2 & 0 & 2 & -3 \end{bmatrix}$$

Solution: By Definition MM,

$$AB = \left[\begin{bmatrix} 2 & 5 \\ -1 & 3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \middle| \begin{bmatrix} 2 & 5 \\ -1 & 3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 5 \\ 0 \end{bmatrix} \middle| \begin{bmatrix} 2 & 5 \\ -1 & 3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} -3 \\ 2 \end{bmatrix} \middle| \begin{bmatrix} 2 & 5 \\ -1 & 3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 4 \\ -2 \end{bmatrix} \right]$$

Repeated applications of Definition MVP give

$$= \left[(1) \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} + (2) \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix} \middle| (5) \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} + (0) \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix} \middle| (-3) \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} + (2) \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix} \middle| (4) \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} + (-3) \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix} \right]$$

$$= \begin{bmatrix} 12 & 10 & 4 & -7 \\ 5 & -5 & 9 & -13 \\ -2 & 10 & -10 & 14 \end{bmatrix}$$

2. Solve the system of equations below using the inverse of a matrix. No credit will be given for solutions obtained with other methods. (15 points)

$$\begin{aligned} x_1 + x_2 + 3x_3 + x_4 &= 5 \\ -2x_1 - x_2 - 4x_3 - x_4 &= -7 \\ x_1 + 4x_2 + 10x_3 + 2x_4 &= 9 \\ -2x_1 - 4x_3 + 5x_4 &= 9 \end{aligned}$$

Solution: The coefficient matrix and vector of constants for the system are

$$\begin{bmatrix} 1 & 1 & 3 & 1 \\ -2 & -1 & -4 & -1 \\ 1 & 4 & 10 & 2 \\ -2 & 0 & -4 & 5 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 5 \\ -7 \\ 9 \\ 9 \end{bmatrix}$$

A^{-1} can be computed by using a calculator, or by the method of Theorem CINSM. Then Theorem SNSCM says the unique solution is

$$A^{-1}\mathbf{b} = \begin{bmatrix} 38 & 18 & -5 & -2 \\ 96 & 47 & -12 & -5 \\ -39 & -19 & 5 & 2 \\ -16 & -8 & 2 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ -7 \\ 9 \\ 9 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 3 \end{bmatrix}$$

3. Let A be the matrix below, and find the indicated sets by the requested methods. (30 points)

$$A = \begin{bmatrix} 2 & -1 & 5 & -3 \\ -5 & 3 & -12 & 7 \\ 1 & 1 & 4 & -3 \end{bmatrix}$$

(a) A linearly independent set S so that $\mathcal{R}(A) = \mathrm{Sp}(S)$ and S is composed of columns of A.

Solution: First find a matrix B that is row-equivalent to A and in reduced row-echelon form

$$B = \begin{bmatrix} \boxed{1} & 0 & 3 & -2 \\ 0 & \boxed{1} & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

By Theorem BROC we can choose the columns of A that correspond to dependent variables ($D = \{1, 2\}$) as the elements of S and obtain the desired properties. So

$$S = \left\{ \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} \right\}$$

(b) A linearly independent set S so that $\mathcal{R}(A) = \mathrm{Sp}(S)$ and the vectors in S have a nice pattern of zeros and ones at the top of the vectors.

Solution: We can write the range of A as the row space of the transpose. So we row-reduce the transpose of A to obtain the row-equivalent matrix C in reduced row-echelon form

$$C = \begin{bmatrix} 1 & 0 & 8 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The nonzero rows (written as columns) will be a linearly independent set that spans the row space of A^t, by Theorem BRS, ans the zeros and ones will be at the top of the vectors,

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 8 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \right\}$$

(c) A linearly independent set S so that $\mathcal{R}(A) = \mathrm{Sp}(S)$ and the vectors in S have a nice pattern of zeros and ones at the bottom of the vectors.

Solution: In preparation for Theorem RNS, augment A with the 3×3 identity matrix I_3 and row-reduce to obtain,

$$\begin{bmatrix} 1 & 0 & 3 & -2 & 0 & -\frac{1}{8} & \frac{3}{8} \\ 0 & 1 & 1 & -1 & 0 & \frac{1}{8} & \frac{5}{8} \\ 0 & 0 & 0 & 0 & 1 & \frac{3}{8} & -\frac{1}{8} \end{bmatrix}$$

Then since the first four columns of row 3 are all zeros, we extract

$$K = \begin{bmatrix} \boxed{1} & \frac{3}{8} & -\frac{1}{8} \end{bmatrix}$$

Theorem RNS says that $\mathcal{R}(A) = \mathcal{N}(K)$. We can then use Theorem SSNS and Theorem BNS to construct the desired set S, based on the free variables with indices in $F = \{2, 3\}$ for the homogeneous system $\mathcal{LS}(K, \mathbf{0})$, so

$$S = \left\{ \begin{bmatrix} -\frac{3}{8} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{8} \\ 0 \\ 1 \end{bmatrix} \right\}$$

Notice that the zeros and ones are at the bottom of the vectors.

(d) A linearly independent set S so that $\mathcal{R}(A) = \mathrm{Sp}(S)$.

Solution: This is a straightforward application of Theorem BRS. Use the row-reduced matrix B from part (a), grab the nonzero rows, and write them as column vectors,

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix} \right\}$$

4. Suppose that A is an $m \times n$ matrix and I_n is the $n \times n$ identity matrix. Give a careful proof that $AI_n = A$. (15 points)

Solution: This is Theorem MMIM and an entry-by-entry proof is given there making use of Theorem EMP. A proof could also be constructed by appealing to Definition MM and then Definition MVP.

5. Suppose that A is an $m \times n$ matrix and B is an $n \times p$ matrix. Prove that the null space of B is a subset of the null space of AB, that is $\mathcal{N}(B) \subseteq \mathcal{N}(AB)$. Provide an example where the opposite is false, in other words give an example where $\mathcal{N}(AB) \not\subseteq \mathcal{N}(B)$. (15 points)

Solution: To prove that one set is a subset of another, we start with an element of the smaller set and see if we can determine that it is a member of the larger set (Technique SE). Suppose $\mathbf{x} \in \mathcal{N}(B)$. Then we know that $B\mathbf{x} = \mathbf{0}$ by Definition NSM. Consider

$$\begin{aligned} (AB)\mathbf{x} &= A(B\mathbf{x}) && \text{Theorem MMA} \\ &= A\mathbf{0} && \text{Hypothesis} \\ &= \mathbf{0} && \text{Theorem MMZM} \end{aligned}$$

To show that the inclusion does not hold in the opposite direction, choose B to be any nonsingular matrix of size n. Then $\mathcal{N}(B) = \{\mathbf{0}\}$ by Theorem NSTNS. Let A be the square zero matrix, $\mathcal{O}$, of the same size. Then $AB = \mathcal{O}B = \mathcal{O}$ by Theorem MMZM and therefore $\mathcal{N}(AB) = \mathbb{C}^n$, and is *not* a subset of $\mathcal{N}(B) = \{\mathbf{0}\}$.