Math 232 Name: Key Dr. Beezer
Quiz D&E Fall 2004

Show *all* of your work and *explain* your answers fully. There is a total of 90 possible points.

1. Doing the computations by hand, find the determinant of the matrix below. Is the matrix singular or nonsingular? (10 points)

$$\begin{bmatrix} 3 & -1 & 4 \\ 2 & 5 & 1 \\ 2 & 0 & 6 \end{bmatrix}$$

Solution: We can expand about any row or column, so the zero entry in the middle of the last row is attractive. Let's expand about column 2. By Theorem DERC you will get the same result by expanding about a different row or column. We will use Theorem DMST twice.

$$\begin{aligned} \begin{vmatrix} 3 & -1 & 4 \\ 2 & 5 & 1 \\ 2 & 0 & 6 \end{vmatrix} &= (-1)(-1)^{1+2}\begin{vmatrix} 2 & 1 \\ 2 & 6 \end{vmatrix} + (5)(-1)^{2+2}\begin{vmatrix} 3 & 4 \\ 2 & 6 \end{vmatrix} + (0)(-1)^{3+2}\begin{vmatrix} 3 & 4 \\ 2 & 1 \end{vmatrix} \\ &= (1)(10) + (5)(10) + 0 = 60 \end{aligned}$$

With a nonzero determinant, Theorem SMZD tells us that the matrix is nonsingular.

2. Doing all of your computations by hand, find the eigenvalues, eigenspaces, algebraic multiplicities and geometric multiplicities for the matrix below. (20 points)

$$B = \begin{bmatrix} -12 & 30 \\ -5 & 13 \end{bmatrix}$$

Solution: The characteristic polynomial of B is

$$\begin{aligned} p_B(x) &= \det(B - xI_2) && \text{Definition CP} \\ &= \begin{vmatrix} -12-x & 30 \\ -5 & 13-x \end{vmatrix} \\ &= (-12-x)(13-x) - (30)(-5) && \text{Theorem DMST} \\ &= x^2 - x - 6 \\ &= (x-3)(x+2) \end{aligned}$$

From this we find eigenvalues $\lambda = 3, -2$ with algebraic multiplicities $\alpha_B(3) = 1$ and $\alpha_B(-2) = 1$.

For eigenvectors and geometric multiplicities, we study the null spaces of $B - \lambda I_2$ (Theorem EMNS).

$\lambda = 3$

$$B - 3I_2 = \begin{bmatrix} -15 & 30 \\ -5 & 10 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 2 \\ 0 & 0 \end{bmatrix}$$

$$E_B(3) = \mathcal{N}(B - 3I_2) = \mathrm{Sp}\left(\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}\right)$$

$\lambda = -2$

$$B + 2I_2 = \begin{bmatrix} -10 & 30 \\ -5 & 15 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & -3 \\ 0 & 0 \end{bmatrix}$$

$$E_B(-2) = \mathcal{N}(B + 2I_2) = \mathrm{Sp}\left(\left\{\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right\}\right)$$

Each eigenspace has dimension one, so we have geometric multiplicities $\gamma_B(3) = 1$ and $\gamma_B(-2) = 1$.

3. Consider the matrix A below. (30 points)

$$A = \begin{bmatrix} 18 & -15 & 33 & -15 \\ -4 & 8 & -6 & 6 \\ -9 & 9 & -16 & 9 \\ 5 & -6 & 9 & -4 \end{bmatrix}$$

(a) $\lambda = 2$ is an eigenvalue of A. Find the algebraic multiplicity of $\lambda = 2$ using your calculator only for row-reducing matrices.

Solution: If $\lambda = 2$ is an eigenvalue of A, the matrix $A - 2I_4$ will be singular, and its null space will be the eigenspace of A. So we form this matrix and row-reduce it,

$$A - 2I_4 = \begin{bmatrix} 16 & -15 & 33 & -15 \\ -4 & 6 & -6 & 6 \\ -9 & 9 & -18 & 9 \\ 5 & -6 & 9 & -6 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 3 & 0 \\ 0 & \boxed{1} & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

With two free variables, we know a basis of the null space (Theorem SSNS and Theorem BNS) will contain two vectors. Thus the null space of $A - 2I_4$ has dimension two, and so the eigenspace of $\lambda = 2$ has dimension two also (Theorem EMNS), $\gamma_A(2) = 2$.

(b) A is diagonalizable. Explain how you can verify this statement without actually doing the diagonalization (you'll do that in part (c)). You may use your calculator without limit on this part.

Solution: Using a calculator, we find that A has three distinct eigenvalues, $\lambda = 3, 2, -1$, with $\lambda = 2$ having algebraic multiplicity two, $\alpha_A(2) = 2$. The eigenvalues $\lambda = 3, -1$ have algebraic multiplicity one, and so by Theorem ME we can conclude that their geometric multiplicities are one as well. Together with the computation of the geometric multiplicity of $\lambda = 2$ from part (a), we know

$$\gamma_A(3) = \alpha_A(3) = 1 \qquad \gamma_A(2) = \alpha_A(2) = 2 \qquad \gamma_A(-1) = \alpha_A(-1) = 1$$

This satisfies the hypotheses of Theorem DMLE, and so we can conclude that A is diagonalizable.

(c) Find a diagonal matrix D and a nonsingular matrix S so that $S^{-1}AS = D$. You may use your calculator without limit on this part.

Solution: A calculator will give us four eigenvectors of A, the two for $\lambda = 2$ being linearly independent hopefully. Or, by hand, we could find basis vectors for the three eigenspaces. For $\lambda = 3, -1$ the eigenspaces have dimension one, and so any eigenvector for these eigenvalues will be multiples of the ones we use below. For $\lambda = 2$ there are many different bases for the eigenspace, so your answer could vary. Our eigenvectors are the basis vectors we would have obtained if we had actually constructed a basis in part (a) rather than just computing the dimension.

By the construction in the proof of Theorem DC, the matrix S has columns that are four linearly independent eigenvectors of A and the diagonal matrix has the eigenvalues on the diagonal (in the same order as the eigenvectors in S). Here are the pieces, "doing" the diagonalization,

$$\begin{bmatrix} -1 & 0 & -3 & 6 \\ -2 & -1 & -1 & 0 \\ 0 & 0 & 1 & -3 \\ 1 & 1 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 18 & -15 & 33 & -15 \\ -4 & 8 & -6 & 6 \\ -9 & 9 & -16 & 9 \\ 5 & -6 & 9 & -4 \end{bmatrix} \begin{bmatrix} -1 & 0 & -3 & 6 \\ -2 & -1 & -1 & 0 \\ 0 & 0 & 1 & -3 \\ 1 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

4. Suppose that A and B are similar matrices. Prove that A^3 and B^3 are similar matrices. (15 points)

   Solution: By Definition SIM we know that there is a nonsingular matrix S so that $A = B^{-1}SB$. Then

$$
\begin{aligned}
A^3 &= (B^{-1}SB)^3 \\
&= (B^{-1}SB)(B^{-1}SB)(B^{-1}SB) \\
&= S^{-1}B(SS^{-1})B(SS^{-1})BS && \text{Theorem MMA} \\
&= S^{-1}B(I_3)B(I_3)BS && \text{Definition MI} \\
&= S^{-1}BBBS && \text{Theorem MMIM} \\
&= S^{-1}B^3S
\end{aligned}
$$

   This equation says that A^3 is similar to B^3 (via the matrix S).

5. A matrix A is idempotent if $A^2 = A$. Show that the only possible eigenvalues of an idempotent matrix are $\lambda = 0$ and $\lambda = 1$. Then give an example of a matrix that is idempotent and has both of these two values as eigenvalues. (15 points)

   Solution: Suppopse that λ is an eigenvalue of A. Then there is an eigenvector $\mathbf{x}$, such that $A\mathbf{x} = \lambda\mathbf{x}$. We have,

$$
\begin{aligned}
\lambda\mathbf{x} &= A\mathbf{x} && \mathbf{x} \text{ eigenvector of } A \\
&= A^2\mathbf{x} && A \text{ is idempotent} \\
&= A(A\mathbf{x}) \\
&= A(\lambda\mathbf{x}) && \mathbf{x} \text{ eigenvector of } A \\
&= \lambda(A\mathbf{x}) && \text{Theorem MMSMM} \\
&= \lambda(\lambda\mathbf{x}) && \mathbf{x} \text{ eigenvector of } A \\
&= \lambda^2\mathbf{x}
\end{aligned}
$$

   From this we get

$$
\begin{aligned}
\mathbf{0} &= \lambda^2\mathbf{x} - \lambda\mathbf{x} \\
&= (\lambda^2 - \lambda)\mathbf{x}
\end{aligned}
$$

   Since $\mathbf{x}$ is an eigenvector, it is nonzero, and Theorem SMEZV leaves us with the conclusion that $\lambda^2 - \lambda = 0$, and the solutions to this quadratic polynomial equation in λ are $\lambda = 0$ and $\lambda = 1$.

   The matrix

$$
\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}
$$

   is idempotent (check this!) and since it is a diagonal matrix, its eigenvalues are the diagonal entries, $\lambda = 0$ and $\lambda = 1$, so each of these possible values for an eigenvalue of an idempotent matrix actually occurs as an eigenvalue of some idempotent matrix.