Math 232 Name: Key Dr. Beezer
Quiz LT Fall 2004

Show *all* of your work and *explain* your answers fully. There is a total of 140 possible points.

For Problems 1–5 consider the linear transformation,

$$T\colon \mathbb{C}^3 \mapsto \mathbb{C}^2, \quad T\left(\begin{bmatrix} x_1\\ x_2\\ x_3\end{bmatrix}\right) = \begin{bmatrix} 2x_1 - x_2 + 5x_3\\ -4x_1 + 2x_2 - 10x_3\end{bmatrix}$$

1. Verify that T is a linear transformation. (15 points)

   Solution: We can rewrite T as follows:

   $$T\left(\begin{bmatrix} x_1\\ x_2\\ x_3\end{bmatrix}\right) = \begin{bmatrix} 2x_1 - x_2 + 5x_3\\ -4x_1 + 2x_2 - 10x_3\end{bmatrix} = x_1\begin{bmatrix} 2\\ -4\end{bmatrix} + x_2\begin{bmatrix} -1\\ 2\end{bmatrix} + x_3\begin{bmatrix} 5\\ -10\end{bmatrix} = \begin{bmatrix} 2 & -1 & 5\\ -4 & 2 & -10\end{bmatrix}\begin{bmatrix} x_1\\ x_2\\ x_3\end{bmatrix}$$

   and Theorem MBLT tell us that any function of this form is a linear transformation.

2. Find a basis for the null space of T, $\mathcal{N}(T)$, and a basis for the range of T, $\mathcal{R}(T)$. (15 points)

   Solution: For the null space, we require all $\mathbf{x} \in \mathbb{C}^3$ such that $T(\mathbf{x}) = \mathbf{0}$. This condition is

   $$\begin{bmatrix} 2x_1 - x_2 + 5x_3\\ -4x_1 + 2x_2 - 10x_3\end{bmatrix} = \begin{bmatrix} 0\\ 0\end{bmatrix}$$

   This leads to a homogeneous system of two linear equations in three variables, whose coefficient matrix row-reduces to

   $$\begin{bmatrix} \boxed{1} & -\frac{1}{2} & \frac{5}{2}\\ 0 & 0 & 0\end{bmatrix}$$

   With two free variables, Theorem SSNS and Theorem BNS yields the basis for the null space

   $$\left\{\begin{bmatrix} -\frac{5}{2}\\ 0\\ 1\end{bmatrix},\ \begin{bmatrix} \frac{1}{2}\\ 1\\ 0\end{bmatrix}\right\}$$

   To find the range of T, apply T to the elements of a spanning set for $\mathbb{C}^3$ as suggested in Theorem SLTS. We will use the standard basis vectors (Theorem SUVB).

   $$\mathcal{R}(T) = \mathrm{Sp}\left(\{T(\mathbf{e}_1), T(\mathbf{e}_2), T(\mathbf{e}_3)\}\right) = \mathrm{Sp}\left(\left\{\begin{bmatrix} 2\\ -4\end{bmatrix},\ \begin{bmatrix} -1\\ 2\end{bmatrix},\ \begin{bmatrix} 5\\ -10\end{bmatrix}\right\}\right)$$

   Each of these vectors is a scalar multiple of the others, so we can toss two of them in reducing the spanning set to a linearly independent set (or be more careful and apply Theorem RBOC on a matrix with these three vectors as columns). The result is the basis of the range,

   $$\left\{\begin{bmatrix} 1\\ -2\end{bmatrix}\right\}$$

3. What is the nullity of T, $n(T)$, and the rank of T, $r(T)$? (10 points)

Solution: The basis for the null space has size 2, so $n(T) = 2$.

The basis for the range has size 1, so $r(T) = 1$.

Check: Confirm Theorem RPNDD,

$$r(T) + n(T) = 2 + 1 = 3 = \dim\left(\mathbb{C}^3\right)$$

4. Is T injective? Surjective? (10 points)

Solution: With $n(T) \neq 0$, $\mathrm{N}(T) \neq \{\mathbf{0}\}$, so Theorem NSILT says T is not injective.

With $r(T) \neq 2$, $\mathrm{R}(T) \neq \mathbb{C}^2$, so Theorem RSLT says T is not surjective.

5. Compute the preimages, $T^{-1}\left(\begin{bmatrix}2\\3\end{bmatrix}\right)$ and $T^{-1}\left(\begin{bmatrix}4\\-8\end{bmatrix}\right)$. (15 points)

Solution: For the first pre-image, we want $\mathbf{x} \in \mathbb{C}^3$ such that $T(\mathbf{x}) = \begin{bmatrix}2\\3\end{bmatrix}$. This becomes,

$$\begin{bmatrix}2x_1 - x_2 + 5x_3\\-4x_1 + 2x_2 - 10x_3\end{bmatrix} = \begin{bmatrix}2\\3\end{bmatrix}$$

Vector equality gives a system of two linear equations in three variables, represented by the augmented matrix

$$\begin{bmatrix}2 & -1 & 5 & 2\\-4 & 2 & -10 & 3\end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix}\boxed{1} & -\frac{1}{2} & \frac{5}{2} & 0\\0 & 0 & 0 & \boxed{1}\end{bmatrix}$$

so the system is inconsistent and the pre-image is the empty set. For the second pre-image the same procedure leads to an augmented matrix with a different vector of constants

$$\begin{bmatrix}2 & -1 & 5 & 4\\-4 & 2 & -10 & -8\end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix}\boxed{1} & -\frac{1}{2} & \frac{5}{2} & 2\\0 & 0 & 0 & 0\end{bmatrix}$$

We begin with just one solution to this system, $\mathbf{x} = \begin{bmatrix}2\\0\\0\end{bmatrix}$, obtained by setting the two free variables (x_2 and x_3) both to zero. Then we apply Theorem NSPI to the non-empty pre-image to get

$$T^{-1}\left(\begin{bmatrix}4\\-8\end{bmatrix}\right) = \mathbf{x} + \mathrm{N}(T) = \begin{bmatrix}2\\0\\0\end{bmatrix} + \mathrm{Sp}\left(\left\{\begin{bmatrix}-\frac{5}{2}\\0\\1\end{bmatrix}, \begin{bmatrix}\frac{1}{2}\\1\\0\end{bmatrix}\right\}\right)$$

6. Consider the linear transformation $S : M_{12} \mapsto P_1$ from the set of 1×2 matrices to the set of polynomials of degree at most 1, defined by

$$S\left(\begin{bmatrix}a & b\end{bmatrix}\right) = (3a + b) + (5a + 2b)x$$

(a) Prove that S is invertible, without using any of your work from part (b). (15 points)

Solution: Determine the null space of S first. The condition that $S\left(\begin{bmatrix}a & b\end{bmatrix}\right) = \mathbf{0}$ becomes $(3a+b)+(5a+2b)x = 0+0x$. Equating coefficients of these polynomials yields the system

$$\begin{aligned} 3a + b &= 0 \\ 5a + 2b &= 0 \end{aligned}$$

This homogeneous system has a nonsingular coefficient matrix, so the only solution is $a = 0$, $b = 0$ and thus

$$\mathrm{N}(S) = \left\{\begin{bmatrix}0 & 0\end{bmatrix}\right\}$$

By Theorem NSILT, we know S is injective. With $n(S) = 0$ we employ Theorem RPNDD to find.

$$r(S) = r(S) + 0 = r(S) + n(S) = \dim(M_{12}) = 2 = \dim(P_1)$$

Since $\mathrm{R}(S) \subseteq P_1$ and $\dim(\mathrm{R}(S)) = \dim(P_1)$, we can infer the set equality $\mathrm{R}(S) = P_1$ and S is surjective. Since S is injective and surjective, S is invertible by Theorem ILTIS.

(b) Perform one-half of the verification that the linear transformation R is the inverse of S. (10 points)

$$R\colon P_1 \mapsto M_{12}, \quad R(r+sx) = \begin{bmatrix}(2r-s) & (-5r+3s)\end{bmatrix}$$

Solution: One of the two defining conditions of an invertible linear transformation is (Definition IVLT)

$$\begin{aligned} (S \circ R)(a+bx) &= S(R(a+bx)) \\ &= S\left(\begin{bmatrix}(2a-b) & (-5a+3b)\end{bmatrix}\right) \\ &= (3(2a-b) + (-5a+3b)) + (5(2a-b) + 2(-5a+3b))\,x \\ &= ((6a-3b) + (-5a+3b)) + ((10a-5b) + (-10a+6b))\,x \\ &= a + bx \\ &= I_{P_1}(a+bx) \end{aligned}$$

That $(R \circ S)\left(\begin{bmatrix}a & b\end{bmatrix}\right) = I_{M_{12}}\left(\begin{bmatrix}a & b\end{bmatrix}\right)$ is similar.

7. Suppose that $T : U \mapsto V$ is a surjective linear transformation and $\dim(U) = \dim(V)$. Prove that T is injective. (20 points)

Solution: If T is surjective, then Theorem RSLT says $\mathrm{R}(T) = V$, so $r(T) = \dim(V)$. In turn, the hypothesis gives $r(T) = \dim(U)$. Then, using Theorem RPNDD,

$$n(T) = (r(T) + n(T)) - r(T) = \dim(U) - \dim(U) = 0$$

With a null space of zero dimension, $\mathrm{N}(T) = \{\mathbf{0}\}$, and by Theorem NSILT we see that T is injective. T is both injective and surjective so by Theorem ILTIS, T is invertible.

8. Suppose that that $T : U \mapsto V$ and $S : V \mapsto W$ are linear transformations. Prove the following relationship between null spaces. (20 points)

$$\mathrm{N}(T) \subseteq \mathrm{N}(S \circ T)$$

Solution: We are asked to prove that $\mathrm{N}(T)$ is a subset of $\mathrm{N}(S \circ T)$. From comments in Technique SE, choose $\mathbf{x} \in \mathrm{N}(T)$. Then we know that $T(\mathbf{x}) = \mathbf{0}$. So

$$\begin{aligned} (S \circ T)(\mathbf{x}) &= S(T(\mathbf{x})) && \text{Definition LTC} \\ &= S(\mathbf{0}) && \mathbf{x} \in \mathrm{N}(T) \\ &= \mathbf{0} && \text{Theorem LTTZZ} \end{aligned}$$

This qualifies $\mathbf{x}$ for membership in $\mathrm{N}(S \circ T)$.