Math 232 | Name: Key | Dr. Beezer

Quiz R | | Fall 2004

Show *all* of your work and *explain* your answers fully. There is a total of 105 possible points.

1. In the vector space $\mathbb{C}^3$, compute the vector representation $\rho_B(\mathbf{v})$ for the basis B and vector $\mathbf{v}$ below. (15 points)

$$B = \left\{ \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix} \right\} \qquad \mathbf{v} = \begin{bmatrix} 11 \\ 5 \\ 8 \end{bmatrix}$$

Solution: We need to express the vector $\mathbf{v}$ as a linear combination of the vectors in B. Theorem VRRB tells us we will be able to do this, and do it uniquely. The vector equation

$$a_1 \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix} + a_2 \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} + a_3 \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 11 \\ 5 \\ 8 \end{bmatrix}$$

becomes (via Theorem SLSLC) a system of linear equations with augmented matrix,

$$\begin{bmatrix} 2 & 1 & 3 & 11 \\ -2 & 3 & 5 & 5 \\ 2 & 1 & 2 & 8 \end{bmatrix}$$

This system has the unique solution $a_1 = 2$, $a_2 = -2$, $a_3 = 3$. So by Definition VR,

$$\rho_B(\mathbf{v}) = \rho_B\left(\begin{bmatrix} 11 \\ 5 \\ 8 \end{bmatrix}\right) = \rho_B\left(2\begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix} + (-2)\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}$$

2. Compute the matrix representation of T relative to the bases B and C. (15 points)

$$T\colon P_3 \mapsto \mathbb{C}^3, \quad T\left(a + bx + cx^2 + dx^3\right) = \begin{bmatrix} 2a - 3b + 4c - 2d \\ a + b - c + d \\ 3a + 2c - 3d \end{bmatrix}$$

$$B = \left\{1,\, x,\, x^2,\, x^3\right\} \qquad C = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Solution: Apply Definition MR,

$$\rho_C(T(1)) = \rho_C\left(\begin{bmatrix}2\\1\\3\end{bmatrix}\right) = \rho_C\left(1\begin{bmatrix}1\\0\\0\end{bmatrix} + (-2)\begin{bmatrix}1\\1\\0\end{bmatrix} + 3\begin{bmatrix}1\\1\\1\end{bmatrix}\right) = \begin{bmatrix}1\\-2\\3\end{bmatrix}$$

$$\rho_C(T(x)) = \rho_C\left(\begin{bmatrix}-3\\1\\0\end{bmatrix}\right) = \rho_C\left((-4)\begin{bmatrix}1\\0\\0\end{bmatrix} + 1\begin{bmatrix}1\\1\\0\end{bmatrix} + 0\begin{bmatrix}1\\1\\1\end{bmatrix}\right) = \begin{bmatrix}-4\\1\\0\end{bmatrix}$$

$$\rho_C\left(T\left(x^2\right)\right) = \rho_C\left(\begin{bmatrix}4\\-1\\2\end{bmatrix}\right) = \rho_C\left(5\begin{bmatrix}1\\0\\0\end{bmatrix} + (-3)\begin{bmatrix}1\\1\\0\end{bmatrix} + 2\begin{bmatrix}1\\1\\1\end{bmatrix}\right) = \begin{bmatrix}5\\-3\\2\end{bmatrix}$$

$$\rho_C\left(T\left(x^3\right)\right) = \rho_C\left(\begin{bmatrix}-2\\1\\-3\end{bmatrix}\right) = \rho_C\left((-3)\begin{bmatrix}1\\0\\0\end{bmatrix} + 4\begin{bmatrix}1\\1\\0\end{bmatrix} + (-3)\begin{bmatrix}1\\1\\1\end{bmatrix}\right) = \begin{bmatrix}-3\\4\\-3\end{bmatrix}$$

These four vectors are the columns of the matrix representation,

$$M^T_{B,C} = \begin{bmatrix}1 & -4 & 5 & -3\\ -2 & 1 & -3 & 4\\ 3 & 0 & 2 & -3\end{bmatrix}$$

3. Find bases for the null space and range of the linear transformation S below. (20 points)

$$S:\, M_{22} \mapsto P_2, \quad S\left(\begin{bmatrix}a & b\\ c & d\end{bmatrix}\right) = (a+2b+5c-4d) + (3a-b+8c+2d)x + (a+b+4c-2d)x^2$$

Solution: These subspaces will be easiest to construct by analyzing a matrix representation of S. Since we can use any matrix representation, we might as well use natural bases that allow us to construct the matrix representation quickly and easily,

$$B = \left\{\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right\} \qquad C = \left\{1,\, x,\, x^2\right\}$$

then we can practically build the matrix representation on sight,

$$M^S_{B,C} = \begin{bmatrix}1 & 2 & 5 & -4\\ 3 & -1 & 8 & 2\\ 1 & 1 & 4 & -2\end{bmatrix}$$

The first step is to find bases for the null space and range of the matrix representation. Row-reducing the matrix representation we find,

$$\begin{bmatrix}1 & 0 & 3 & 0\\ 0 & 1 & 1 & -2\\ 0 & 0 & 0 & 0\end{bmatrix}$$

So by Theorem SSNS, Theorem BNS and Theorem BROC, we have

$$\mathrm{N}\left(M^S_{B,C}\right) = \mathrm{Sp}\left(\left\{\begin{bmatrix}-3\\-1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\2\\0\\1\end{bmatrix}\right\}\right) \qquad \mathrm{R}\left(M^S_{B,C}\right) = \mathrm{Sp}\left(\left\{\begin{bmatrix}1\\3\\1\end{bmatrix}, \begin{bmatrix}2\\-1\\1\end{bmatrix}\right\}\right)$$

Now, the proofs of Theorem INS and Theorem IR tell us that we can apply ρ_B^{-1} and ρ_C^{-1} (respectively) to "un-coordinatize" and get bases for the null space and range of the linear transformation S itself,

$$\mathrm{N}(S) = \mathrm{Sp}\left(\left\{\begin{bmatrix} -3 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}\right\}\right) \qquad \mathrm{R}(S) = \mathrm{Sp}\left(\left\{1 + 3x + x^2,\, 2 - x + x^2\right\}\right)$$

4. Let S_{22} be the set of 2×2 symmetric matrices. Verify that the linear transformation R is invertible and find R^{-1}. (20 points)

$$R:\, S_{22} \mapsto P_2, \quad R\left(\begin{bmatrix} a & b \\ b & c \end{bmatrix}\right) = (a-b) + (2a - 3b - 2c)x + (a - b + c)x^2$$

Solution: The analysis of R will be easiest if we analyze a matrix representation of R. Since we can use any matrix representation, we might as well use natural bases that allow us to construct the matrix representation quickly and easily,

$$B = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\} \qquad C = \left\{1,\, x,\, x^2\right\}$$

then we can practically build the matrix representation on sight,

$$M^R_{B,C} = \begin{bmatrix} 1 & -1 & 0 \\ 2 & -3 & -2 \\ 1 & -1 & 1 \end{bmatrix}$$

This matrix representation is invertible (it has a nonzero determinant of -1, Theorem SMZD, Theorem NSI) so Theorem IMR tells us that the linear transformation S is also invertible. To find a formula for R^{-1} we compute,

$$\begin{aligned}
R^{-1}\left(a + bx + cx^2\right) &= \rho_B^{-1}\left(M^{R^{-1}}_{B,C}\,\rho_C\left(a + bx + cx^2\right)\right) && \text{Theorem FTMR} \\
&= \rho_B^{-1}\left(\left(M^R_{B,C}\right)^{-1}\rho_C\left(a + bx + cx^2\right)\right) && \text{Theorem IMR} \\
&= \rho_B^{-1}\left(\left(M^R_{B,C}\right)^{-1}\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) && \text{Definition VR} \\
&= \rho_B^{-1}\left(\begin{bmatrix} 5 & -1 & -2 \\ 4 & -1 & -2 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) && \text{Definition MI} \\
&= \rho_B^{-1}\left(\begin{bmatrix} 5a - b - 2c \\ 4a - b - 2c \\ -a + c \end{bmatrix}\right) && \text{Definition MVP} \\
&= \begin{bmatrix} 5a - b - 2c & 4a - b - 2c \\ 4a - b - 2c & -a + c \end{bmatrix} && \text{Definition VR}
\end{aligned}$$

5. Find a basis for the vector space P_3 composed of eigenvectors of the linear transformation T. Find a matrix representation of T relative to this basis. (20 points)

$$T:\, P_3 \mapsto P_3, \quad T\left(a + bx + cx^2 + dx^3\right) = (a + c + d) + (b + c + d)x + (a + b + c)x^2 + (a + b + d)x^3$$

Solution: With the domain and codomain being identical, we will build a matrix representation using the same basis for both the domain and codomain. The eigenvalues of the matrix representation will be the eigenvalues of the linear transformation, and we can obtain the eigenvectors of the linear transformation by un-coordinatizing (Theorem EER). Since the method does not depend on *which* basis we choose, we can choose a natural basis for ease of computation, say,

$$B = \left\{1,\, x,\, x^2, x^3\right\}$$

The matrix representation is then,

$$M^T_{B,B} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

The eigenvalues and eigenvectors of this matrix were computed in Example EE.ESMS4. A basis for $\mathbb{C}^4$, composed of eigenvectors of the matrix representation is,

$$C = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Applying ρ_B^{-1} to each vector of this set, yields a basis of P_3 composed of eigenvectors of T,

$$D = \left\{1 + x + x^2 + x^3, -1 + x,\, -x^2 + x^3,\, -1 - x + x^2 + x^3\right\}$$

The matrix representation of T relative to the basis D will be a diagonal matrix with the corresponding eigenvalues along the diagonal, so in this case we get

$$M^T_{D,D} = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

6. Suppose that $T : V \mapsto V$ is an invertible linear transformation with a nonzero eigenvalue λ. Prove that $\dfrac{1}{\lambda}$ is an eigenvalue of T^{-1}. (15 points)

Solution: Let $\mathbf{v}$ be an eigenvector of T for the eigenvalue λ. Then,

$$\begin{aligned}
T^{-1}(\mathbf{v}) &= \frac{1}{\lambda}\lambda T^{-1}(\mathbf{v}) && \lambda \neq 0 \\
&= \frac{1}{\lambda} T^{-1}(\lambda\mathbf{v}) && \text{Theorem ILTLT} \\
&= \frac{1}{\lambda} T^{-1}(T(\mathbf{v})) && \mathbf{v}\text{ eigenvector of } T \\
&= \frac{1}{\lambda} I_V(\mathbf{v}) && \text{Definition IVLT} \\
&= \frac{1}{\lambda}\mathbf{v} && \text{Definition IDLT}
\end{aligned}$$

which says that $\dfrac{1}{\lambda}$ is an eigenvalue of T^{-1} with eigenvector $\mathbf{v}$. Note that it is possible to prove that any eigenvalue of an invertible linear transformation is never zero. So the hypothesis that λ be nonzero is just a convenience for this problem.