Math 232 Name: Key Dr. Beezer
Quiz V Fall 2005

Show *all* of your work and *explain* your answers fully. There is a total of 95 possible points. If you use a calculator on a problem be sure to write down both the input to, and output from, the calculator.

1. Determine if the set S below is linearly independent or linearly dependent. (15 points)

$$S = \left\{ \begin{bmatrix} 2 \\ 1 \\ 3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \\ 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 10 \\ -7 \\ 0 \\ 10 \\ 4 \end{bmatrix} \right\}$$

Solution: Theorem LIVRN suggests we analyze a matrix whose columns are the vectors from the set,

$$A = \begin{bmatrix} 2 & 4 & 10 \\ 1 & -2 & -7 \\ 3 & 1 & 0 \\ -1 & 3 & 10 \\ 2 & 2 & 4 \end{bmatrix}$$

Row-reducing the matrix A yields,

$$\begin{bmatrix} \boxed{1} & 0 & -1 \\ 0 & \boxed{1} & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

We see that $r = 2 \neq 3 = n$, where r is the number of nonzero rows and n is the number of columns. By Theorem LIVRN, the set S is linearly dependent.

2. Let S be the set of vectors below from $\mathbb{C}^3$. (20 points)

$$S = \left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix}, \begin{bmatrix} -6 \\ 5 \\ 1 \end{bmatrix} \right\}$$

(a) Determine if the vector $\mathbf{y} = \begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix}$ is an element of $\mathcal{S}p(S)$.

Solution: Form a linear combination, with unknown scalars, of S that equals $\mathbf{y}$,

$$a_1 \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} + a_2 \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} + a_3 \begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix} + a_4 \begin{bmatrix} -6 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix}$$

We want to know if there are values for the scalars that make the vector equation true since that is the definition of membership in $\mathcal{S}p(S)$. By Theorem SLSLC any such values will also be solutions to the linear system represented by the augmented matrix,

$$\begin{bmatrix} -1 & 3 & 1 & -6 & -5 \\ 2 & 1 & 5 & 5 & 3 \\ 1 & 2 & 4 & 1 & 0 \end{bmatrix}$$

Row-reducing the matrix yields,

$$\begin{bmatrix} \boxed{1} & 0 & 2 & 3 & 2 \\ 0 & \boxed{1} & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

From this we see that the system of equations is consistent (Theorem RCLS), and has a infinitely many solutions. Any solution will provide a linear combination of the vectors in R that equals $\mathbf{y}$. So $\mathbf{y} \in R$, for example,

$$(-10)\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} + (-2)\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} + (3)\begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix} + (2)\begin{bmatrix} -6 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix}$$

(b) Determine if the vector $\mathbf{w} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$ is an element of $\mathcal{S}p(S)$.

Solution: Form a linear combination, with unknown scalars, of S that equals $\mathbf{w}$,

$$a_1\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} + a_2\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} + a_3\begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix} + a_4\begin{bmatrix} -6 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

We want to know if there are values for the scalars that make the vector equation true since that is the definition of membership in $\mathcal{S}p(S)$. By Theorem SLSLC any such values will also be solutions to the linear system represented by the augmented matrix,

$$\begin{bmatrix} -1 & 3 & 1 & -6 & 2 \\ 2 & 1 & 5 & 5 & 1 \\ 1 & 2 & 4 & 1 & 3 \end{bmatrix}$$

Row-reducing the matrix yields,

$$\begin{bmatrix} \boxed{1} & 0 & 2 & 3 & 0 \\ 0 & \boxed{1} & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \end{bmatrix}$$

With a leading 1 in the last column, the system is inconsistent (Theorem RCLS), so there are no scalars a_1, a_2, a_3, a_4 that will create a linear combination of the vectors in S that equal $\mathbf{w}$. So $\mathbf{w} \notin \mathcal{S}p(S)$.

3. For the matrix A below, find a linearly independent set S so that the null space of T is spanned by S, that is, $\mathcal{N}(A) = \mathcal{S}p(S)$. (15 points)

$$A = \begin{bmatrix} -1 & -2 & 2 & 1 & 5 \\ 1 & 2 & 1 & 1 & 5 \\ 3 & 6 & 1 & 2 & 7 \\ 2 & 4 & 0 & 1 & 2 \end{bmatrix}$$

Solution: Theorem BNS provides formulas for $n-r$ vectors that will meet the requirements of this question. These vectors are the same ones listed in Theorem VFSLS when we solve the homogeneous system $\mathcal{LS}(A, \mathbf{0})$, whose solution set is the null space (Definition NSM).

To apply Theorem BNS or Theorem VFSLS we first row-reduce the matrix, resulting in

$$B = \begin{bmatrix} \boxed{1} & 2 & 0 & 0 & 3 \\ 0 & 0 & \boxed{1} & 0 & 6 \\ 0 & 0 & 0 & \boxed{1} & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

So we see that $n - r = 5 - 3 = 2$ and $F = \{2, 5\}$, so the vector form of a generic solution vector is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ -6 \\ 4 \\ 1 \end{bmatrix}$$

So we have

$$\mathcal{N}(A) = \mathcal{S}p\left(\left\{\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -6 \\ 4 \\ 1 \end{bmatrix}\right\}\right)$$

4. Given the set S below, find a linearly independent set T so that $\mathcal{S}p(T) = \mathcal{S}p(S)$. (15 points)

$$S = \left\{\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 5 \\ -1 \\ 3 \end{bmatrix}\right\}$$

Solution: Theorem RSS says we can make a matrix with these four vectors as columns, row-reduce, and just keep the columns with indices in the set D. Here we go, forming the relevant matrix and row-reducing,

$$\begin{bmatrix} 2 & 3 & 1 & 5 \\ -1 & 0 & 1 & -1 \\ 2 & 1 & -1 & 3 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & -1 & 1 \\ 0 & \boxed{1} & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Analyzing the row-reduced version of this matrix, we see that the firast two columns are pivot columns, so $D = \{1, 2\}$. Theorem RSS says we need only "keep" the first two columns to create a set with the requisite properties,

$$T = \left\{\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}\right\}$$

5. Suppose that $\mathbf{x}$ is a solution to $\mathcal{LS}(A,\, \mathbf{b})$ and that $\mathbf{z}$ is a solution to the homogeneous system $\mathcal{LS}(A,\, \mathbf{0})$. Prove that $\mathbf{x}+\mathbf{z}$ is a solution to $\mathcal{LS}(A,\, \mathbf{b})$. (15 points)

   Solution: Suppose that A has n columns, so $\mathbf{x},\, \mathbf{z} \in \mathbb{C}^n$. Give the components of $\mathbf{x}$ and $\mathbf{z}$ names and apply Definition CVA,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} \qquad \mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ \vdots \\ z_n \end{bmatrix} \qquad \mathbf{x}+\mathbf{z} = \begin{bmatrix} x_1+z_1 \\ x_2+z_2 \\ x_3+z_3 \\ \vdots \\ x_n+z_n \end{bmatrix}$$

   We wish to prove that the latter vector is a solution to $\mathcal{LS}(A,\, \mathbf{b})$ on the assumption that $\mathbf{x}$ and $\mathbf{z}$ are solutions to the given systems. Suppose that a single equation from this system (the i-th one) has the form,

$$a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 + \cdots + a_{in}x_n = b_i$$

   Evaluate the left-hand side of this equation with the components of the proposed solution vector $\mathbf{x}+\mathbf{z}$,

$$\begin{aligned}
&a_{i1}(x_1+z_1) + a_{i2}(x_2+z_2) + a_{i3}(x_3+z_3) + \cdots + a_{in}(x_n+z_n) \\
&\quad = a_{i1}x_1 + a_{i1}z_1 + a_{i2}x_2 + a_{i2}z_2 + a_{i3}x_3 + a_{i3}z_3 + \cdots + a_{in}x_n + a_{in}z_n && \text{Distributivity} \\
&\quad = a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 + \cdots + a_{in}x_n + a_{i1}z_1 + a_{i2}z_2 + a_{i3}z_3 + \cdots + a_{in}z_n && \text{Commutativity} \\
&\quad = a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 + \cdots + a_{in}x_n + 0 && \mathbf{z} \text{ solution to } \mathcal{LS}(A,\, \mathbf{0}) \\
&\quad = b_i + 0 && \mathbf{x} \text{ solution to } \mathcal{LS}(A,\, \mathbf{b}) \\
&\quad = b_i
\end{aligned}$$

   So equation i is true for all i, and we see that $\mathbf{x}+\mathbf{z}$ is a solution to $\mathcal{LS}(A,\, \mathbf{b})$.

6. Suppose that $\mathbf{v}_1$ and $\mathbf{v}_2$ are any two vectors from $\mathbb{C}^m$. Prove the following set equality. (15 points)

$$\mathcal{S}p(\{\mathbf{v}_1,\, \mathbf{v}_2\}) = \mathcal{S}p(\{\mathbf{v}_1+\mathbf{v}_2,\, \mathbf{v}_1-\mathbf{v}_2\})$$

   Solution: This is an equality of sets, so Technique SE applies.

   The "easy" half first. Show that $X = \mathcal{S}p(\{\mathbf{v}_1+\mathbf{v}_2,\, \mathbf{v}_1-\mathbf{v}_2\}) \subseteq \mathcal{S}p(\{\mathbf{v}_1,\, \mathbf{v}_2\}) = Y$.
   Choose $\mathbf{x} \in X$. Then $\mathbf{x} = a_1(\mathbf{v}_1+\mathbf{v}_2) + a_2(\mathbf{v}_1-\mathbf{v}_2)$ for some scalars a_1 and a_2. Then,

$$\begin{aligned}
\mathbf{x} &= a_1(\mathbf{v}_1+\mathbf{v}_2) + a_2(\mathbf{v}_1-\mathbf{v}_2) \\
&= a_1\mathbf{v}_1 + a_1\mathbf{v}_2 + a_2\mathbf{v}_1 + (-a_2)\mathbf{v}_2 \\
&= (a_1+a_2)\mathbf{v}_1 + (a_1-a_2)\mathbf{v}_2
\end{aligned}$$

   which qualifies $\mathbf{x}$ for membership in Y, as it is a linear combination of $\mathbf{v}_1,\, \mathbf{v}_2$.

   Now show the opposite inclusion, $Y = \mathcal{S}p(\{\mathbf{v}_1,\, \mathbf{v}_2\}) \subseteq \mathcal{S}p(\{\mathbf{v}_1+\mathbf{v}_2,\, \mathbf{v}_1-\mathbf{v}_2\}) = X$.
   Choose $\mathbf{y} \in Y$. Then there are scalars $b_1,\, b_2$ such that $\mathbf{y} = b_1\mathbf{v}_1 + b_2\mathbf{v}_2$. Rearranging, we obtain,

$$\begin{aligned}
\mathbf{y} &= b_1\mathbf{v}_1 + b_2\mathbf{v}_2 \\
&= \frac{b_1}{2}\left[(\mathbf{v}_1+\mathbf{v}_2) + (\mathbf{v}_1-\mathbf{v}_2)\right] + \frac{b_2}{2}\left[(\mathbf{v}_1+\mathbf{v}_2) - (\mathbf{v}_1-\mathbf{v}_2)\right] \\
&= \frac{b_1+b_2}{2}(\mathbf{v}_1+\mathbf{v}_2) + \frac{b_1-b_2}{2}(\mathbf{v}_1-\mathbf{v}_2)
\end{aligned}$$

   This is an expression for $\mathbf{y}$ as a linear combination of $\mathbf{v}_1+\mathbf{v}_2$ and $\mathbf{v}_1-\mathbf{v}_2$, earning $\mathbf{y}$ membership in X. Since X is a subset of Y, and vice versa, we see that $X = Y$, as desired.