Math 232 Name: Key Dr. Beezer
Quiz M Fall 2005

Show *all* of your work and *explain* your answers fully. There is a total of 100 possible points. If you use a calculator on a problem be sure to write down both the input to, and output from, the calculator.

1. Use the inverse of a matrix to find all the solutions to the following system of equations. No credit will be given for solutions found by another method. (15 points)

$$\begin{aligned} x_1 + 2x_2 - x_3 &= -3 \\ 2x_1 + 5x_2 - x_3 &= -4 \\ -x_1 - 4x_2 &= 2 \end{aligned}$$

Solution: The coefficient matrix of this system of equations is

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 5 & -1 \\ -1 & -4 & 0 \end{bmatrix}$$

and the vector of constants is $\mathbf{b} = \begin{bmatrix} -3 \\ -4 \\ 2 \end{bmatrix}$. So by Theorem SLEMM we can convert the system to the form $A\mathbf{x} = \mathbf{b}$. Row-reducing this matrix yields the identity matrix so by Theorem NSRRI we know A is nonsingular. This allows us to apply Theorem SNSCM to find the unique solution as

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} -4 & 4 & 3 \\ 1 & -1 & -1 \\ -3 & 2 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ -4 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

Remember, you can check this solution easily by evaluating the matrix-vector product $A\mathbf{x}$ (Definition MVP).

2. For the matrix A below find a set of vectors T meeting the following requirements: (1) the span of T is the column space of A, that is, $\mathcal{S}p(T) = \mathcal{C}(A)$, (2) T is linearly independent, and (3) the elements of T are columns of A. (15 points)

$$A = \begin{bmatrix} 2 & 1 & 4 & -1 & 2 \\ 1 & -1 & 5 & 1 & 1 \\ -1 & 2 & -7 & 0 & 1 \\ 2 & -1 & 8 & -1 & 2 \end{bmatrix}$$

Solution: Theorem BROC is the right tool for this problem. Row-reduce this matrix, identify the pivot columns and then grab the corresponding columns of A for the set T. The matrix A row-reduces to

$$\begin{bmatrix} \boxed{1} & 0 & 3 & 0 & 0 \\ 0 & \boxed{1} & -2 & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \end{bmatrix}$$

So $D = \{1,\,2,\,4,\,5\}$ and then

$$T = \{\mathbf{A}_1,\,\mathbf{A}_2,\,\mathbf{A}_3,\,\mathbf{A}_4\} = \left\{ \begin{bmatrix} 2 \\ 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \\ 2 \end{bmatrix} \right\}$$

has the requested properties.

3. Given the matrix A below, use the extended echelon form of A to answer each part of this problem. No credit will be given for other approaches that do not use the extended echelon form. In each part, find a linearly independent set of vectors, S, so that the span of S, $\mathcal{S}p(S)$, equals the specified set of vectors. (40 points)

$$A = \begin{bmatrix} -5 & 3 & -1 \\ -1 & 1 & 1 \\ -8 & 5 & -1 \\ 3 & -2 & 0 \end{bmatrix}$$

Solution: Add a 4×4 identity matrix to the right of A to form the matrix M and then row-reduce to the matrix N,

$$M = \begin{bmatrix} -5 & 3 & -1 & 1 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & 1 & 0 & 0 \\ -8 & 5 & -1 & 0 & 0 & 1 & 0 \\ 3 & -2 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 2 & 0 & 0 & -2 & -5 \\ 0 & \boxed{1} & 3 & 0 & 0 & -3 & -8 \\ 0 & 0 & 0 & \boxed{1} & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & \boxed{1} & 1 & 3 \end{bmatrix} = N$$

To apply Theorem FS in each of these four parts, we need the two matrices,

$$C = \begin{bmatrix} \boxed{1} & 0 & 2 \\ 0 & \boxed{1} & 3 \end{bmatrix} \qquad L = \begin{bmatrix} \boxed{1} & 0 & -1 & -1 \\ 0 & \boxed{1} & 1 & 3 \end{bmatrix}$$

(a) The row space of A, $\mathcal{R}(A)$.

Solution:

$$\begin{aligned} \mathcal{R}(A) &= \mathcal{R}(C) && \text{Theorem FS} \\ &= \mathcal{S}p\left(\left\{\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}\right\}\right) && \text{Theorem BRS} \end{aligned}$$

(b) The column space of A, $\mathcal{C}(A)$

Solution:

$$\begin{aligned} \mathcal{C}(A) &= \mathcal{N}(L) && \text{Theorem FS} \\ &= \mathcal{S}p\left(\left\{\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 0 \\ 1 \end{bmatrix}\right\}\right) && \text{Theorem BNS} \end{aligned}$$

(c) The null space of A, $\mathcal{N}(A)$.

Solution:

$$\begin{aligned} \mathcal{N}(A) &= \mathcal{N}(C) && \text{Theorem FS} \\ &= \mathcal{S}p\left(\left\{\begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix}\right\}\right) && \text{Theorem BNS} \end{aligned}$$

(d) The left null space of A, $\mathcal{L}(A)$.

Solution:

$$\begin{aligned}
\mathcal{L}(A) &= \mathcal{R}(L) && \text{Theorem FS} \\
&= \mathcal{S}p\left(\begin{bmatrix}1\\0\\-1\\-1\end{bmatrix},\ \begin{bmatrix}0\\1\\1\\3\end{bmatrix}\right) && \text{Theorem BRS}
\end{aligned}$$

4. Suppose that A is an $m \times n$ matrix and let I_n denote the $n \times n$ identity matrix. Prove that $AI_n = A$. (15 points)

Solution: This is the first part of Theorem MMIM. See the proof given there.

5. Suppose that A is an $m \times n$ matrix and B is a $n \times p$ matrix. Prove that the null space of B is a subset of the null space of AB, that is $\mathcal{N}(B) \subseteq \mathcal{N}(AB)$. (15 points)

Solution: This is Exercise MM.T40. See the proof given there.