Math 232 Name: Key Dr. Beezer
Quiz VS Fall 2005

Show *all* of your work and *explain* your answers fully. There is a total of 90 possible points. If you use a calculator on a problem be sure to write down both the input to, and output from, the calculator.

1. In the vector space of polynomials $P_3$, determine if the set $S$ is linearly independent or linearly dependent. (15 points)

$$S = \left\{2 + x - 3x^2 - 8x^3,\ 1 + x + x^2 + 5x^3,\ 3 - 4x^2 - 7x^3\right\}$$

Solution: Begin with a relation of linear dependence (Definition RLD),

$$a_1\left(2 + x - 3x^2 - 8x^3\right) + a_2\left(1 + x + x^2 + 5x^3\right) + a_3\left(3 - 4x^2 - 7x^3\right) = \mathbf{0}$$

Massage according to the definitions of scalar multiplication and vector addition in the definition of $P_3$ (Example VS.VSP) and use the zero vector dro this vector space,

$$(2a_1 + a_2 + 3a_3) + (a_1 + a_2)\,x + (-3a_1 + a_2 - 4a_3)\,x^2 + (-8a_1 + 5a_2 - 7a_3)\,x^3 = 0 + 0x + 0x^2 + 0x^3$$

The definition of the equality of polynomials allows us to deduce the following four equations,

$$\begin{aligned} 2a_1 + a_2 + 3a_3 &= 0 \\ a_1 + a_2 &= 0 \\ -3a_1 + a_2 - 4a_3 &= 0 \\ -8a_1 + 5a_2 - 7a_3 &= 0 \end{aligned}$$

Row-reducing the coefficient matrix of this homogeneous system leads to the unique solution $a_1 = a_2 = a_3 = 0$. So the only relation of linear dependence on $S$ is the trivial one, and this is linear independence for $S$ (Definition LI).

2. The set $W$ is a subspace of $M_{22}$, the vector space of all $2 \times 2$ matrices. Prove that $S$ is a spanning set for $W$. (15 points)

$$W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \,\middle|\, 2a - 3b + 4c - d = 0 \right\} \qquad S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & -4 \end{bmatrix} \right\}$$

Solution: We want to show that $W = \mathcal{S}p(S)$ (Definition TSVS), which is an equality of sets (Technique SE).

First, show that $\mathcal{S}p(S) \subseteq W$. Begin by checking that each of the three matrices in $S$ is a member of the set $W$. Then, since $W$ is a vector space, the closure properties (Property AC, Property SC) guarantee that every linear combination of elements of $S$ remains in $W$.

Second, show that $W \subseteq \mathcal{S}p(S)$. We want to convince ourselves that an arbitrary element of $W$ is a linear combination of elements of $S$. Choose

$$\mathbf{x} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in W$$

The values of $a$, $b$, $c$, $d$ are not totally arbitary, since membership in $W$ requires that $2a - 3b + 4c - d = 0$. Now, rewrite as follows,

$$
\begin{aligned}
\mathbf{x} &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\
&= \begin{bmatrix} a & b \\ c & 2a - 3b + 4c \end{bmatrix} && 2a - 3b + 4c - d = 0 \\
&= \begin{bmatrix} a & 0 \\ 0 & 2a \end{bmatrix} + \begin{bmatrix} 0 & b \\ 0 & -3b \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ c & 4c \end{bmatrix} && \text{Definition MA} \\
&= a \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & -3 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 4 \end{bmatrix} && \text{Definition MSM} \\
&\in \mathcal{S}p(S) && \text{Definition SS}
\end{aligned}
$$

3. For the matrix $A$ below, compute the dimension of the null space of $A$, $\dim(\mathcal{N}(A))$. (15 points)

$$
A = \begin{bmatrix} 2 & -1 & -3 & 11 & 9 \\ 1 & 2 & 1 & -7 & -3 \\ 3 & 1 & -3 & 6 & 8 \\ 2 & 1 & 2 & -5 & -3 \end{bmatrix}
$$

Solution: Row reduce $A$,

$$
A \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 0 & 1 & 1 \\ 0 & \boxed{1} & 0 & -3 & -1 \\ 0 & 0 & \boxed{1} & -2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}
$$

So $r = 3$ for this matrix. Then

$$
\begin{aligned}
\dim(\mathcal{N}(A)) &= n(A) && \text{Definition NOM} \\
&= (n(A) + r(A)) - r(A) \\
&= 5 - r(A) && \text{Theorem RPNC} \\
&= 5 - 3 && \text{Theorem CRN} \\
&= 2
\end{aligned}
$$

We could also use Theorem BNS and create a basis for $\mathcal{N}(A)$ with $n - r = 5 - 3 = 2$ vectors (because the solutions are described with 2 free variables) and arrive at the dimension as the size of this basis.

4. The set $W$ below is a subspace of $\mathbb{C}^4$. Find the dimension of $W$. (15 points)

$$
W = \mathcal{S}p\left(\left\{ \begin{bmatrix} 2 \\ -3 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -4 \\ -3 \\ 2 \\ 5 \end{bmatrix} \right\}\right)
$$

Solution: We will appeal to Theorem RSS (or you could consider this an appeal to Theorem BCSOC). Put the three columnn vectors of this spanning set into a matrix as columns and row-reduce.

$$
\begin{bmatrix} 2 & 3 & -4 \\ -3 & 0 & -3 \\ 4 & 1 & 2 \\ 1 & -2 & 5 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 1 \\ 0 & \boxed{1} & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
$$

The pivot columns are $D = \{1, 2\}$ so we can "keep" the vectors corresponding to the pivot columns and set

$$T = \left\{ \begin{bmatrix} 2 \\ -3 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \\ -2 \end{bmatrix} \right\}$$

and conclude that $W = \mathcal{S}p(T)$ and $T$ is linearly independent. In other words, $T$ is a basis with two vectors, so $W$ has dimension 2.

5. The set $Z = \left\{ \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \middle| \; 4z_1 - 3z_2 + z_3 = 0 \right\}$ is a subset of $\mathbb{C}^3$. Use the three-part test of Theorem TSS to prove that $Z$ is a subspace of $\mathbb{C}^3$. No credit will be given for using other methods. (15 points)

   Solution: See the solution for Exercise S.M20 which is almost identical. This question asks for the "less direct" solution.

6. Suppose that $V$ is a vector space and $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_m\}$ is a subset of $V$. Prove that $\mathcal{S}p(S)$ is a subspace of $V$. (15 points)

   Solution: This is Theorem SSS. See the proof given there.