Math 232 | Name: Key | Dr. Beezer
Quiz D&E | | Fall 2005

Show *all* of your work and *explain* your answers fully. There is a total of 90 possible points. If you use a calculator on a problem be sure to write down both the input to, and output from, the calculator.

1. Compute the determinant of the matrix B by hand, in other words, without using a calculator. (10 points)

$$B = \begin{bmatrix} -2 & 3 & -2 \\ -4 & -2 & 1 \\ 2 & 4 & 2 \end{bmatrix}$$

Solution: We'll expand about the first row since there are no zeros to exploit,

$$\begin{aligned} \det(B) &= \begin{vmatrix} -2 & 3 & -2 \\ -4 & -2 & 1 \\ 2 & 4 & 2 \end{vmatrix} \\ &= (-2)\begin{vmatrix} -2 & 1 \\ 4 & 2 \end{vmatrix} + (-1)(3)\begin{vmatrix} -4 & 1 \\ 2 & 2 \end{vmatrix} + (-2)\begin{vmatrix} -4 & -2 \\ 2 & 4 \end{vmatrix} \\ &= (-2)((-2)(2) - (1)(4)) + (-3)((-4)(2) - (1)(2)) + (-2)((-4)(4) - (-2)(2)) \\ &= (-2)(-8) + (-3)(-10) + (-2)(-12) \\ &= 70 \end{aligned}$$

2. Find all of the eigenvalues and eigenspaces for the matrix C by hand, in other words, without using a calculator. (15 points)

$$C = \begin{bmatrix} -1 & 2 \\ -6 & 6 \end{bmatrix}$$

Solution: First compute the characteristic polynomial,

$$\begin{aligned} p_C(x) &= \det(C - xI_2) && \text{Definition CP} \\ &= \begin{vmatrix} -1-x & 2 \\ -6 & 6-x \end{vmatrix} && = (-1-x)(6-x) - (2)(-6) \\ &= x^2 - 5x - 6 \\ &= (x-3)(x-2) \end{aligned}$$

So the eigenvalues of C are the solutions to $p_C(x) = 0$, namely, $\lambda = 2$ and $\lambda = 3$.

To obtain the eigenspaces, construct the appropriate singular matrices and find expressions for the null spaces of these matrices.

$$\lambda = 2$$

$$C - (2)I_2 = \begin{bmatrix} -3 & 2 \\ -6 & 4 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & -\frac{2}{3} \\ 0 & 0 \end{bmatrix}$$

$$E_C(2) = \mathcal{N}(C - (2)I_2) = \mathcal{S}p\left(\left\{\begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}\right\}\right) = \mathcal{S}p\left(\left\{\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right\}\right)$$

$$\lambda = 3$$

$$C - (3)I_2 = \begin{bmatrix} -4 & 2 \\ -6 & 3 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & -\frac{1}{2} \\ 0 & 0 \end{bmatrix}$$

$$E_C\left(3\right) = \mathcal{N}(C - (3)I_2) = \mathcal{S}p\left(\left\{\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}\right\}\right) = \mathcal{S}p\left(\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}\right)$$

3. Consider the matrix A. (35 points)

$$A = \begin{bmatrix} 19 & 25 & 30 & 5 \\ -23 & -30 & -35 & -5 \\ 7 & 9 & 10 & 1 \\ -3 & -4 & -5 & -1 \end{bmatrix}$$

(a) Find the eigenvalues of A using a calculator and use these to construct the characteristic polynomial of A, $p_A(x)$. State the algebraic multiplicity of each eigenvalue.

Solution: A calculator will report $\lambda = 0$ as an eigenvalue of algebraic multiplicity of 2, and $\lambda = -1$ as an eigenvalue of algebraic multiplicity 2 as well. Since eigenvalues are roots of the characteristic polynomial (Theorem EMRCP) we have the factored version

$$p_A(x) = (x-0)^2(x-(-1))^2 = x^2(x^2+2x+1) = x^4 + 2x^3 + x^2$$

(b) Find all of the eigenspaces for A by computing expressions for null spaces. Only use your calculator to row-reduce matrices. State the geometric multiplicity of each eigenvalue.

Solution:

$$\lambda = 0$$

$$A - (0)I_4 = \begin{bmatrix} 19 & 25 & 30 & 5 \\ -23 & -30 & -35 & -5 \\ 7 & 9 & 10 & 1 \\ -3 & -4 & -5 & -1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & -5 & -5 \\ 0 & \boxed{1} & 5 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$E_A(0) = \mathcal{N}(C - (0)I_4) = \mathcal{S}p\left(\left\{ \begin{bmatrix} 5 \\ -5 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ -4 \\ 0 \\ 1 \end{bmatrix} \right\}\right)$$

$$\lambda = -1$$

$$A - (-1)I_4 = \begin{bmatrix} 20 & 25 & 30 & 5 \\ -23 & -29 & -35 & -5 \\ 7 & 9 & 11 & 1 \\ -3 & -4 & -5 & 0 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & -1 & 4 \\ 0 & \boxed{1} & 2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$E_A(-1) = \mathcal{N}(C - (-1)I_4) = \mathcal{S}p\left(\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \\ 1 \end{bmatrix} \right\}\right)$$

Each eigenspace above is described by a spanning set obtained through an application of Theorem BNS and so is a basis for the eigenspace. In each case the dimension, and therefore the geometric multiplicity, is 2.

(c) Is A diagonalizable? If not, explain why. If so, find a diagonal matrix D that is similar to A.

Solution: For each of the two eigenvalues, the algebraic and geometric multiplicities are equal. Theorem DMLE says that in this situation the matrix is diagonalizable. We know from Theorem DC that when we diagonalize A the diagonal matrix will have the eigenvalues of A on the diagonal (in some order). So we can claim that

$$D = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

4. Can a single vector be an eigenvector of a matrix for two different eigenvalues? Give a proof or counterexample to justify your answer. (15 points)

   Solution: No.

   Suppose that the vector $\mathbf{x}$ is an eigenvector of A for the eigenvalues λ and ρ. Then

$$\begin{aligned}
\mathbf{0} &= A\mathbf{x} - A\mathbf{x} \\
&= \lambda\mathbf{x} - \rho\mathbf{x} && \lambda,\ \rho, \text{ eigenvalues of } A \\
&= (\lambda - \rho)\,\mathbf{x} && \text{Property DSAC}
\end{aligned}$$

   Now apply Theorem SMEZV to conclude that $(\lambda - \rho) = 0$ or $\mathbf{x} = \mathbf{0}$. The second possibility is false, since eigenvectors are never the zero vector (Definition EEM). So $(\lambda - \rho) = 0$, in other words, $\lambda = \rho$. So we see that a single vector cannot serve two masters.

5. Suppose that A is a square matrix. Prove that the constant term of the characteristic polynomial of A is equal to the determinant of A. (15 points)

   Solution: Suppose that the characteristic polynomial of A is

$$p_A(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$$

   Then

$$\begin{aligned}
a_0 &= a_0 + a_1(0) + a_2(0)^2 + \cdots + a_n(0)^n \\
&= p_A(0) \\
&= \det(A - 0I_n) && \text{Definition CP} \\
&= \det(A)
\end{aligned}$$