Math 232 Name: Key Dr. Beezer
Quiz LT Fall 2005

Show *all* of your work and *explain* your answers fully. There is a total of 90 possible points. If you use a calculator on a problem be sure to write down both the input to, and output from, the calculator.

1. The linear transformation $T\colon \mathbb{C}^4 \mapsto \mathbb{C}^3$ is not injective. Find two inputs $\mathbf{x},\,\mathbf{y} \in \mathbb{C}^4$ that yield the same output (that is $T(\mathbf{x}) = T(\mathbf{y})$.) (15 points)

$$T\left(\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}\right) = \begin{bmatrix}2x_1 + x_2 + x_3\\ -x_1 + 3x_2 + x_3 - x_4\\ 3x_1 + x_2 + 2x_3 - 2x_4\end{bmatrix}$$

Solution: A linear transformation that is not injective will have a non-trivial kernel (Theorem KILT), and this is the key to finding the desired inputs. We need one non-trivial element of the kernel, so suppose that $\mathbf{z} \in \mathbb{C}^4$ is an element of the kernel,

$$\begin{bmatrix}0\\0\\0\end{bmatrix} = \mathbf{0} = T(\mathbf{z}) = \begin{bmatrix}2z_1 + z_2 + z_3\\ -z_1 + 3z_2 + z_3 - z_4\\ 3z_1 + z_2 + 2z_3 - 2z_4\end{bmatrix}$$

Vector equality Definition CVE leads to the homogeneous system of three equations in four variables,

$$\begin{aligned} 2z_1 + z_2 + z_3 &= 0\\ -z_1 + 3z_2 + z_3 - z_4 &= 0\\ 3z_1 + z_2 + 2z_3 - 2z_4 &= 0 \end{aligned}$$

The coefficient matrix of this system row-reduces as

$$\begin{bmatrix}2 & 1 & 1 & 0\\ -1 & 3 & 1 & -1\\ 3 & 1 & 2 & -2\end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix}\boxed{1} & 0 & 0 & 1\\ 0 & \boxed{1} & 0 & 1\\ 0 & 0 & \boxed{1} & -3\end{bmatrix}$$

From this we can find a solution (we only need one), that is an element of $\mathcal{K}(T)$,

$$\mathbf{z} = \begin{bmatrix}-1\\-1\\3\\1\end{bmatrix}$$

Now, we choose a vector $\mathbf{x}$ at random and set $\mathbf{y} = \mathbf{x} + \mathbf{z}$,

$$\mathbf{x} = \begin{bmatrix}2\\3\\4\\-2\end{bmatrix} \qquad \mathbf{y} = \mathbf{x} + \mathbf{z} = \begin{bmatrix}2\\3\\4\\-2\end{bmatrix} + \begin{bmatrix}-1\\-1\\3\\1\end{bmatrix} = \begin{bmatrix}1\\2\\7\\-1\end{bmatrix}$$

and you can check that

$$T(\mathbf{x}) = \begin{bmatrix}11\\13\\21\end{bmatrix} = T(\mathbf{y})$$

2. The linear transformation $S\colon \mathbb{C}^4 \mapsto \mathbb{C}^3$ is not surjective. Find an output $\mathbf{w} \in \mathbb{C}^3$ that has an empty pre-image (that is $S^{-1}(\mathbf{w}) = \emptyset$.) (15 points)

$$S\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 + x_2 + 3x_3 - 4x_4 \\ x_1 + 3x_2 + 4x_3 + 3x_4 \\ -x_1 + 2x_2 + x_3 + 7x_4 \end{bmatrix}$$

Solution: To find an element of $\mathbb{C}^3$ with an empty pre-image, we will compute the range of the linear transformation $\mathcal{R}(S)$ and then find an element outside of this set.

By Theorem SSRLT we can evaluate S with the elements of a spanning set of the domain and create a spanning set for the range.

$$S\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \qquad S\left(\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \qquad S\left(\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} \qquad S\left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -4 \\ 3 \\ 7 \end{bmatrix}$$

So

$$\mathcal{R}(S) = \mathcal{S}p\left(\left\{\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} -4 \\ 3 \\ 7 \end{bmatrix}\right\}\right)$$

This spanning set is obviously linearly dependent, so we can reduce it to a basis for $\mathcal{R}(S)$ using Theorem BRS, where the elements of the spanning set are placed as the rows of a matrix. The result is that

$$\mathcal{R}(S) = \mathcal{S}p\left(\left\{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right\}\right)$$

Therefore, the unique vector in $\mathcal{R}(S)$ with a first slot equal to 6 and a second slot equal to 15 will be the linear combination

$$6\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + 15\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 15 \\ 9 \end{bmatrix}$$

So, any vector with first two components equal to 6 and 15, but with a third component different from 9, such as

$$\mathbf{w} = \begin{bmatrix} 6 \\ 15 \\ -63 \end{bmatrix}$$

will not be an element of the range of S and will therefore have an empty pre-image.

3. Determine if the linear transformation $T\colon P_2 \mapsto M_{22}$ is (a) injective, (b) surjective, (c) invertible. (15 points)

$$T\left(a + bx + cx^2\right) = \begin{bmatrix} a + 2b - 2c & 2a + 2b \\ -a + b - 4c & 3a + 2b + 2c \end{bmatrix}$$

Solution: (a) We will compute the kernel of T. Suppose that $a + bx + cx^2 \in \mathcal{K}(T)$. Then

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = T\left(a + bx + cx^2\right) = \begin{bmatrix} a + 2b - 2c & 2a + 2b \\ -a + b - 4c & 3a + 2b + 2c \end{bmatrix}$$

and matrix equality (Theorem ME) yields the homogeneous system of four equations in three variables,

$$
\begin{aligned}
a + 2b - 2c &= 0 \\
2a + 2b &= 0 \\
-a + b - 4c &= 0 \\
3a + 2b + 2c &= 0
\end{aligned}
$$

The coefficient matrix of this system row-reduces as

$$
\begin{bmatrix} 1 & 2 & -2 \\ 2 & 2 & 0 \\ -1 & 1 & -4 \\ 3 & 2 & 2 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 2 \\ 0 & \boxed{1} & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
$$

From the existence of non-trivial solutions to this system, we can infer non-zero polynomials in $\mathcal{K}(T)$. By Theorem KILT we then know that T is not injective.

(b) Since $3 = \dim(P_2) < \dim(M_{22}) = 4$, by Theorem SLTD T is not surjective.

(c) Since T is not surjective, it is not invertible by Theorem ILTIS.

4. Determine if the linear transformation $S\colon P_3 \mapsto M_{22}$ is (a) injective, (b) surjective, (c) invertible. (15 points)

$$
T\left(a + bx + cx^2 + dx^3\right) = \begin{bmatrix} -a + 4b + c + 2d & 4a - b + 6c - d \\ a + 5b - 2c + 2d & a + 2c + 5d \end{bmatrix}
$$

Solution: (a) To check injectivity, we compute the kernel of S. To this end, suppose that $a+bx+cx^2+dx^3 \in \mathcal{K}(S)$, so

$$
\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = S\left(a + bx + cx^2 + dx^3\right) = \begin{bmatrix} -a + 4b + c + 2d & 4a - b + 6c - d \\ a + 5b - 2c + 2d & a + 2c + 5d \end{bmatrix}
$$

this creates the homogeneous system of four equations in four variables,

$$
\begin{aligned}
-a + 4b + c + 2d &= 0 \\
4a - b + 6c - d &= 0 \\
a + 5b - 2c + 2d &= 0 \\
a + 2c + 5d &= 0
\end{aligned}
$$

The coefficient matrix of this system row-reduces as,

$$
\begin{bmatrix} -1 & 4 & 1 & 2 \\ 4 & -1 & 6 & -1 \\ 1 & 5 & -2 & 2 \\ 1 & 0 & 2 & 5 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & \boxed{1} \end{bmatrix}
$$

We recognize the coefficient matrix as being nonsingular, so the only solution to the system is $a = b = c = d = 0$, and the kernel of S is trivial, $\mathcal{K}(S) = \left\{0 + 0x + 0x^2 + 0x^3\right\}$. By Theorem KILT, we see that S is injective.

(b) We can establish that S is surjective by considering the rank and nullity of S.

$$
\begin{aligned}
r(S) &= \dim(P_3) - n(S) && \text{Theorem RPNDD} \\
&= 4 - 0 \\
&= \dim(M_{22})
\end{aligned}
$$

So, $\mathcal{R}(S)$ is a subspace of M_{22} (Theorem RLTS) whose dimension equals that of M_{22}. By Theorem EDYES, we gain the set equality $\mathcal{R}(S) = M_{22}$. Theorem RSLT then implies that S is surjective.

(c) Since S is both injective and surjective, Theorem ILTIS says S is invertible.

5. The linear transformation $R\colon M_{21} \mapsto M_{12}$ is invertible. Determine a formula for the inverse linear transformation $R^{-1}\colon M_{12} \mapsto M_{21}$. (15 points)

$$R\left(\begin{bmatrix} a & b \end{bmatrix}\right) = \begin{bmatrix} a+3b \\ 4a+11b \end{bmatrix}$$

Solution: We are given that R is invertible. The inverse linear transformation can be formulated by considering the pre-image of a generic element of the codomain. With injectivity and surjectivity, we know that the pre-image of any element will be a set of size one — it is this lone element that will be the output of the inverse linear transformation.

Suppose that we set $\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ as a generic element of the codomain, M_{21}. Then if $\begin{bmatrix} r & s \end{bmatrix} = \mathbf{w} \in R^{-1}(\mathbf{v})$,

$$\begin{aligned} \begin{bmatrix} x \\ y \end{bmatrix} &= \mathbf{v} = R(\mathbf{w}) \\ &= \begin{bmatrix} r+3s \\ 4r+11s \end{bmatrix} \end{aligned}$$

So we obtain the system of two equations in the two variables r and s,

$$\begin{aligned} r + 3s &= x \\ 4r + 11s &= y \end{aligned}$$

With a nonsingular coefficient matrix, we can solve the system using the inverse of the coefficient matrix,

$$\begin{aligned} r &= 11x + 3y \\ s &= 4x - y \end{aligned}$$

So we define,

$$R^{-1}(\mathbf{v}) = R^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \mathbf{w} = \begin{bmatrix} r & s \end{bmatrix} = \begin{bmatrix} 11x+3y \\ 4x-y \end{bmatrix}$$

6. Suppose that $T\colon U \mapsto V$ and $S\colon V \mapsto U$ are linear transformations. Prove that $\mathcal{K}(T) \subseteq \mathcal{K}(S \circ T)$. (15 points)

Solution: This is Exercise ILT.T15, see the proof there.