Math 232 Name: Key Dr. Beezer
Quiz V Spring 2005

Show *all* of your work and *explain* your answers fully. There is a total of 100 possible points.

1. Let $S = \left\{ \begin{bmatrix} 1 \\ -2 \\ 2 \\ 5 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 1 \\ 2 \\ -4 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \\ 2 \end{bmatrix} \right\}$. Is S linearly independent? (10 points)

Solution: Theorem LIVRN suggests we analyze a matrix whose columns are the vectors of S,

$$A = \begin{bmatrix} 1 & 3 & 2 & 1 \\ -2 & 3 & 1 & 0 \\ 2 & 1 & 2 & 1 \\ 5 & 2 & -1 & 2 \\ 3 & -4 & 1 & 2 \end{bmatrix}$$

Row-reducing the matrix A yields,

$$\begin{bmatrix} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We see that $r = 4 = n$, where r is the number of nonzero rows and n is the number of columns. By Theorem LIVRN, the set S is linearly independent.

2. Let $T = \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ -1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 4 \\ -2 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -1 \\ -2 \\ 0 \end{bmatrix} \right\}$. Is T linearly independent? (10 points)

Solution: Theorem LIVRN suggests we analyze a matrix whose columns are the vectors of S,

$$A = \begin{bmatrix} 1 & 3 & 4 & -1 \\ 2 & 2 & 4 & 2 \\ -1 & -1 & -2 & -1 \\ 0 & 2 & 2 & -2 \\ 1 & 2 & 3 & 0 \end{bmatrix}$$

Row-reducing the matrix A yields,

$$\begin{bmatrix} \boxed{1} & 0 & 1 & 2 \\ 0 & \boxed{1} & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We see that $r = 2 \neq 4 = n$, where r is the number of nonzero rows and n is the number of columns. By Theorem LIVRN, the set S is not linearly independent.

3. Suppose $R = \left\{ \begin{bmatrix} 2 \\ -1 \\ 3 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 0 \\ 3 \\ -2 \end{bmatrix} \right\}$.

(a) Is $\mathbf{y} = \begin{bmatrix} 1 \\ -1 \\ -8 \\ -4 \\ -3 \end{bmatrix}$ in $\mathcal{S}p(R)$? (10 points)

Solution: Form a linear combination, with unknown scalars, of R that equals $\mathbf{y}$,

$$a_1 \begin{bmatrix} 2 \\ -1 \\ 3 \\ 4 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \\ -1 \end{bmatrix} + a_3 \begin{bmatrix} 3 \\ -1 \\ 0 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -8 \\ -4 \\ -3 \end{bmatrix}$$

We want to know if there are values for the scalars that make the vector equation true since that is the definition of membership in $\mathcal{S}p(R)$. By Theorem SLSLC any such values will also be solutions to the linear system represented by the augmented matrix,

$$\begin{bmatrix} 2 & 1 & 3 & 1 \\ -1 & 1 & -1 & -1 \\ 3 & 2 & 0 & -8 \\ 4 & 2 & 3 & -4 \\ 0 & -1 & -2 & -3 \end{bmatrix}$$

Row-reducing the matrix yields,

$$\begin{bmatrix} \boxed{1} & 0 & 0 & -2 \\ 0 & \boxed{1} & 0 & -1 \\ 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

From this we see that the system of equations is consistent (Theorem RCLS), and has a unique solution. This solution will provide a linear combination of the vectors in R that equals $\mathbf{y}$. So $\mathbf{y} \in R$.

(b) Is $\mathbf{z} = \begin{bmatrix} 1 \\ 1 \\ 5 \\ 3 \\ 1 \end{bmatrix}$ in $\mathcal{S}p(R)$? (10 points)

Solution: Form a linear combination, with unknown scalars, of R that equals $\mathbf{z}$,

$$a_1 \begin{bmatrix} 2 \\ -1 \\ 3 \\ 4 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \\ -1 \end{bmatrix} + a_3 \begin{bmatrix} 3 \\ -1 \\ 0 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 5 \\ 3 \\ 1 \end{bmatrix}$$

We want to know if there are values for the scalars that make the vector equation true since that is the definition of membership in $\mathcal{S}p(R)$. By Theorem SLSLC any such values will also be solutions to the linear system represented by the augmented matrix,

$$\begin{bmatrix} 2 & 1 & 3 & 1 \\ -1 & 1 & -1 & 1 \\ 3 & 2 & 0 & 5 \\ 4 & 2 & 3 & 3 \\ 0 & -1 & -2 & 1 \end{bmatrix}$$

Row-reducing the matrix yields,

$$\begin{bmatrix} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

With a leading 1 in the last column, the system is inconsistent (Theorem RCLS), so there are no scalars a_1, a_2, a_3 that will create a linear combination of the vectors in R that equal $\mathbf{z}$. So $\mathbf{z} \notin R$.

4. For the matrix B below, find a set S that is linearly independent and spans the null space of B, that is, $\mathcal{N}(B) = \mathcal{S}p(S)$. (15 points)

$$B = \begin{bmatrix} -3 & 1 & -2 & 7 \\ -1 & 2 & 1 & 4 \\ 1 & 1 & 2 & -1 \end{bmatrix}$$

Solution: The requested set is described by Theorem BNS. It is easiest to find by using the procedure of Example VFSAL. Begin by row-reducing the matrix, viewing it as the coefficient matrix of a homogeneous system of equations. We obtain,

$$\begin{bmatrix} \boxed{1} & 0 & 1 & -2 \\ 0 & \boxed{1} & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Now build the vector form of the solutions to this homogeneous system (Theorem VFSLS). The free variables are x_3 and x_4, corresponding to the columns without leading 1's,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

The desired set S is simply the constant vectors in this expression, and these are the vectors $\mathbf{z}_1$ and $\mathbf{z}_2$ described by Theorem BNS.

$$S = \left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

5. Let T be the set of columns of the matrix B from the problem above. Define $W = \mathcal{S}p(T)$. Find a set R so that (1) R has 3 vectors, (2) R is a subset of T, and (3) $W = \mathcal{S}p(R)$. (15 points)

Solution: Let $T = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4\}$. The second vector of the set S from the previous problem is a solution to the homogeneous system with the matrix B as the coefficient matrix. By Theorem SLSLC it provides the scalars for a linear combination of the columns of B (the vectors in T) that equals the zero vector, a relation of linear dependence on T,

$$2\mathbf{w}_1 + (-1)\mathbf{w}_2 + (1)\mathbf{w}_4 = \mathbf{0}$$

We can rearrange this equation by solving for $\mathbf{w}_4$,

$$\mathbf{w}_4 = (-2)\mathbf{w}_1 + \mathbf{w}_2$$

This equation tells us that the vector $\mathbf{w}_4$ is superfluous in the span construction that creates W. So $W = \mathcal{S}p(\{\mathbf{w}_1,\, \mathbf{w}_2,\, \mathbf{w}_3\})$. The requested set is $R = \{\mathbf{w}_1,\, \mathbf{w}_2,\, \mathbf{w}_3\}$.

6. Suppose that $\mathbf{u},\, \mathbf{v},\, \mathbf{w} \in \mathbb{C}^m$. Prove that $\langle \mathbf{u} + \mathbf{v},\, \mathbf{w}\rangle = \langle \mathbf{u},\, \mathbf{w}\rangle + \langle \mathbf{v},\, \mathbf{w}\rangle$. (15 points)

   Solution:

$$\begin{aligned}
\langle \mathbf{u} + \mathbf{v},\, \mathbf{w}\rangle &= \sum_{i=1}^{m} (u_i + v_i)(\overline{w_i}) && \text{Definition IP} \\
&= \sum_{i=1}^{m} u_i\overline{w_i} + v_i\overline{w_i} && \text{Distributivity in } \mathbb{C} \\
&= \sum_{i=1}^{m} u_i\overline{w_i} + \sum_{i=1}^{m} v_i\overline{w_i} && \text{Commutativity in } \mathbb{C} \\
&= \langle \mathbf{u},\, \mathbf{w}\rangle + \langle \mathbf{v},\, \mathbf{w}\rangle && \text{Definition IP}
\end{aligned}$$

7. Suppose that $\mathbf{v}_1,\, \mathbf{v}_2 \in \mathbb{C}^m$. Prove the following. (15 points)

$$\mathcal{S}p(\{\mathbf{v}_1,\, \mathbf{v}_2\}) = \mathcal{S}p(\{\mathbf{v}_1,\, \mathbf{v}_2,\, 5\mathbf{v}_1 + 3\mathbf{v}_2\})$$

   Solution: This is an equality of sets, so Technique SE applies.

   $X = \mathcal{S}p(\{\mathbf{v}_1,\, \mathbf{v}_2\}) \subseteq \mathcal{S}p(\{\mathbf{v}_1,\, \mathbf{v}_2,\, 5\mathbf{v}_1 + 3\mathbf{v}_2\}) = Y$
   Choose $\mathbf{x} \in X$. Then $\mathbf{x} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2$ for some scalars a_1 and a_2. Then,

$$\mathbf{x} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + 0(5\mathbf{v}_1 + 3\mathbf{v}_2)$$

   which qualifies $\mathbf{x}$ for membership in Y, as it is a linear combination of $\mathbf{v}_1,\, \mathbf{v}_2,\, 5\mathbf{v}_1 + 3\mathbf{v}_2$.

   $Y = \mathcal{S}p(\{\mathbf{v}_1,\, \mathbf{v}_2,\, 5\mathbf{v}_1 + 3\mathbf{v}_2\}) \subseteq \mathcal{S}p(\{\mathbf{v}_1,\, \mathbf{v}_2\}) = X$
   Choose $\mathbf{y} \in Y$. Then there are scalars $a_1,\, a_2,\, a_3$ such that

$$\mathbf{y} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3(5\mathbf{v}_1 + 3\mathbf{v}_2)$$

   Rearranging, we obtain,

$$\begin{aligned}
\mathbf{y} &= a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3(5\mathbf{v}_1 + 3\mathbf{v}_2) \\
&= a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + 5a_3\mathbf{v}_1 + 3a_3\mathbf{v}_2 && \text{Property DVAC} \\
&= a_1\mathbf{v}_1 + 5a_3\mathbf{v}_1 + a_2\mathbf{v}_2 + 3a_3\mathbf{v}_2 && \text{Property CC} \\
&= (a_1 + 5a_3)\mathbf{v}_1 + (a_2 + 3a_3)\mathbf{v}_2 && \text{Property DSAC}
\end{aligned}$$

   this is an expression for $\mathbf{y}$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$, earning $\mathbf{y}$ membership in X. Since X is a subset of Y, and vice versa, we see that $X = Y$, as desired.