Math 232 Name: Key Dr. Beezer
Quiz M Spring 2005

Show *all* of your work and *explain* your answers fully. There is a total of 100 possible points.

1. Find the inverse of the matrix C with your calculator, but limit yourself to the `rref` command. (10 points)

$$C = \begin{bmatrix} 1 & 1 & 3 & 1 \\ -2 & -1 & -4 & -1 \\ 1 & 4 & 10 & 2 \\ -2 & 0 & -4 & 5 \end{bmatrix}$$

Solution: Employ Theorem CINSM,

$$\begin{bmatrix} 1 & 1 & 3 & 1 & 1 & 0 & 0 & 0 \\ -2 & -1 & -4 & -1 & 0 & 1 & 0 & 0 \\ 1 & 4 & 10 & 2 & 0 & 0 & 1 & 0 \\ -2 & 0 & -4 & 5 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 0 & 0 & 38 & 18 & -5 & -2 \\ 0 & \boxed{1} & 0 & 0 & 96 & 47 & -12 & -5 \\ 0 & 0 & \boxed{1} & 0 & -39 & -19 & 5 & 2 \\ 0 & 0 & 0 & \boxed{1} & -16 & -8 & 2 & 1 \end{bmatrix}$$

And therefore we see that C is nonsingular (it row-reduces to the identity matrix, Theorem NSRRI) and by Theorem CINSM,

$$C^{-1} = \begin{bmatrix} 38 & 18 & -5 & -2 \\ 96 & 47 & -12 & -5 \\ -39 & -19 & 5 & 2 \\ -16 & -8 & 2 & 1 \end{bmatrix}$$

2. Find all solutions to the system of equations below, employing the inverse found in the previous problem. No credit will be given for solution sets found using other techniques. (10 points)

$$\begin{aligned} x_1 + x_2 + 3x_3 + x_4 &= -4 \\ -2x_1 - x_2 - 4x_3 - x_4 &= 4 \\ x_1 + 4x_2 + 10x_3 + 2x_4 &= -20 \\ -2x_1 - 4x_3 + 5x_4 &= 9 \end{aligned}$$

Solution: View this system as $\mathcal{LS}(C, \mathbf{b})$, where $\mathbf{b} = \begin{bmatrix} -4 \\ 4 \\ -20 \\ 9 \end{bmatrix}$. Since C was seen to be nonsingular in the previous problem Theorem SNSCM says the solution, which is unique by Theorem NSMUS, is given by

$$C^{-1}\mathbf{b} = \begin{bmatrix} 38 & 18 & -5 & -2 \\ 96 & 47 & -12 & -5 \\ -39 & -19 & 5 & 2 \\ -16 & -8 & 2 & 1 \end{bmatrix} \begin{bmatrix} -4 \\ 4 \\ -20 \\ 9 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -2 \\ 1 \end{bmatrix}$$

3. Construct an example of a 4×4 orthogonal matrix. (10 points)

Solution: The 4×4 identity matrix, I_4, would be one example (Definition IM). Any of the 23 other rearrangements of the columns of I_4 would be a simple, but less trivial, example. See Example OMP.

4. For the matrix B below, find sets of vectors whose span equals the range ($\mathcal{R}(B)$)and which meet the indicated requirements of each part. (30 points)

$$B = \begin{bmatrix} 2 & 3 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ -1 & 2 & 3 & -4 \end{bmatrix}$$

(a) The set illustrates the definition of the range.

Solution: The definition of the range is the span of the set of columns (Definition RM). So the desired set is just the four columns of B,

$$S = \left\{ \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -4 \end{bmatrix} \right\}$$

(b) The set is linearly independent and the members of the set are columns of B.

Solution: Theorem BROC suggests row-reducing the matrix and using the pivot columns.

$$B \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & -1 & 2 \\ 0 & \boxed{1} & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So the pivot columns are numbered by elements of $D = \{1, 2\}$, so the requested set is

$$S = \left\{ \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \right\}$$

(c) The set is linearly independent with a "nice pattern of zeros and ones" at the top of each vector.

Solution: We can find this set by row-reducing the transpose of B, deleting the zero rows, and using the nonzero rows as column vectors in the set. This is an application of Theorem RMRST followed by Theorem BRS.

$$B^t \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 3 \\ 0 & \boxed{1} & -7 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So the requested set is

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -7 \end{bmatrix} \right\}$$

(d) The set is linearly independent with a "nice pattern of zeros and ones" at the bottom of each vector.

Solution: With the range expressed as null space, the vectors obtained via Theorem BNS will be of the desired shape. So we first proceed with Theorem RNS,

$$[B \mid I_3] = \begin{bmatrix} 2 & 3 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ -1 & 2 & 3 & -4 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & -1 & 2 & 0 & \frac{2}{3} & \frac{-1}{3} \\ 0 & \boxed{1} & 1 & -1 & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & \boxed{1} & \frac{-7}{3} & \frac{-1}{3} \end{bmatrix}$$

So, employing Theorem RNS, we have $\mathcal{R}(B) = \mathcal{N}(K)$, where

$$K = \begin{bmatrix} \boxed{1} & \frac{-7}{3} & \frac{-1}{3} \end{bmatrix}$$

We can find the desired set of vectors from Theorem BNS as

$$S = \left\{ \begin{bmatrix} \frac{7}{3} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{3} \\ 0 \\ 1 \end{bmatrix} \right\}$$

5. Find a linearly independent set S so that the row space of the matrix B (see previous problem) is the span of S ($\mathcal{S}p(S) = \mathcal{RS}(B)$), and S is linearly independent. (10 points)

Solution: Theorem BRS is the most direct route to set with these properties. Row-reduce, toss zero rows, keep the others. You could transpose the matrix, then look for the range by row-reducing the transpose and applying Theorem BROC. We'll do the former,

$$B \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & -1 & 2 \\ 0 & \boxed{1} & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So the set S is

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix} \right\}$$

6. For the matrix E below, find vectors $\mathbf{b}$ and $\mathbf{c}$ so that the system $\mathcal{LS}(E, \mathbf{b})$ is consistent and $\mathcal{LS}(E, \mathbf{c})$ is inconsistent. (15 points)

$$E = \begin{bmatrix} -2 & 1 & 1 & 0 \\ 3 & -1 & 0 & 2 \\ 4 & 1 & 1 & 6 \end{bmatrix}$$

Solution: Any vector from $\mathbb{C}^3$ will lead to a consistent system, and therefore there is no vector that will lead to an inconsistent system. How do we convince ourselves of this? First, row-reduce E,

$$E \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 0 & 1 \\ 0 & \boxed{1} & 0 & 1 \\ 0 & 0 & \boxed{1} & 1 \end{bmatrix}$$

If we augment E with any vector of constants, and row-reduce the augmented matrix, we will never find a leading 1 in the final column, so by Theorem RCLS the system will always be consistent.

Another approach is to apply Theorem RNS and since E has no zero rows once row-reduced, we reach the conclusion that the range is all of $\mathbb{C}^3$, $\mathcal{R}(E) = \mathbb{C}^3$. Then Theorem RCS then says that any system with E as a coefficient matrix will be consistent.

7. Suppose that A is an $m \times n$ matrix, and $\alpha \in \mathbb{C}$ is a scalar. Give a careful proof, providing justifications for each step, that $(\alpha A)^t = \alpha A^t$. (15 points)

Solution: This is Theorem TMSM. See the proof given there.