Math 232 Name: Key Dr. Beezer
Quiz VS Spring 2005

Show *all* of your work and *explain* your answers fully. There is a total of 90 possible points.

1. Consider the subspace

$$W = \mathcal{S}p\left(\left\{\begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix}, \begin{bmatrix} 4 & 0 \\ 2 & 3 \end{bmatrix}, \begin{bmatrix} -3 & 1 \\ 2 & 1 \end{bmatrix}\right\}\right)$$

of the vector space of 2×2 matrices, M_{22}. Is $C = \begin{bmatrix} -3 & 3 \\ 6 & -4 \end{bmatrix}$ an element of W? (15 points)

Solution: In order to belong to W, we must be able to express C as a linear combination of the elements in the spanning set of W. So we begin with such an expression, using the unknowns a, b, c for the scalars in the linear combination.

$$C = \begin{bmatrix} -3 & 3 \\ 6 & -4 \end{bmatrix} = a\begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix} + b\begin{bmatrix} 4 & 0 \\ 2 & 3 \end{bmatrix} + c\begin{bmatrix} -3 & 1 \\ 2 & 1 \end{bmatrix}$$

Massaging the right-hand side, according to the definition of the vector space operations in M_{22} (Example VSM), we find the matrix equality,

$$\begin{bmatrix} -3 & 3 \\ 6 & -4 \end{bmatrix} = \begin{bmatrix} 2a + 4b - 3c & a + c \\ 3a + 2b + 2c & -a + 3b + c \end{bmatrix}$$

Matrix equality allows us to form a system of four equations in three variables, whose augmented matrix row-reduces as follows,

$$\begin{bmatrix} 2 & 4 & -3 & -3 \\ 1 & 0 & 1 & 3 \\ 3 & 2 & 2 & 6 \\ -1 & 3 & 1 & -4 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 0 & 2 \\ 0 & \boxed{1} & 0 & -1 \\ 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since this system of equations is consistent (Theorem RCLS), a solution will provide values for a, b and c that alllow us to recognize C as an element of W.

2. Determine if the set $T = \left\{x^2 - x + 5,\ 4x^3 - x^2 + 5x,\ 3x + 2\right\}$ spans the vector space of polynomials with degree 4 or less, P_4. (15 points)

Solution: The vector space P_4 has dimension 5 by Theorem DP. Since T contains only 3 vectors, and $3 < 5$, Theorem G tells us that T does not span P_5.

3. In the crazy vector space C (Example CVS), is the set $S = \{(0,\, 2),\ (2,\, 8)\}$ linearly independent? (15 points)

Solution: We begin with a relation of linear dependence using unknown scalars a and b. We wish to know if these scalars *must* both be zero. Recall that the zero vector in C is $(-1,\, -1)$ and that the definitions of vector addition and scalar multiplication are not what we might expect.

$$\begin{aligned}
\mathbf{0} = (-1,\, -1) &= a(0,\, 2) + b(2,\, 8) && \text{Definition RLD}\\
&= (0a + a - 1,\, 2a + a - 1) + (2b + b - 1,\, 8b + b - 1) && \text{Scalar multiplication, Example CVS}\\
&= (a - 1,\, 3a - 1) + (3b - 1,\, 9b - 1)\\
&= (a - 1 + 3b - 1 + 1,\, 3a - 1 + 9b - 1 + 1) && \text{Vector addition, Example CVS}\\
&= (a + 3b - 1,\, 3a + 9b - 1)
\end{aligned}$$

From this we obtain two equalities, which can be converted to a homogeneous system of equations,

$$\begin{aligned}
-1 &= a + 3b - 1 & \qquad a + 3b &= 0\\
-1 &= 3a + 9b - 1 & \qquad 3a + 9b &= 0
\end{aligned}$$

This homogeneous system has a singular coefficient matrix (Theorem SMZD), and so has more than just the trivial solution (Definition NM). Any nontrivial solution will give us a nontrivial relation of linear dependence on S. So S is linearly dependent (Definition LI).

4. A 2×2 matrix B is upper-triangular if $[B]_{21} = 0$. Let UT_2 be the set of all 2×2 upper-triangular matrices. Then UT_2 is a subspace of the vector space of all 2×2 matrices, M_{22} (you may assume this). Determine the dimension of UT_2 providing *all* of the necessary justifications for your answer. (15 points)

Solution: A typical matrix from UT_2 looks like

$$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$$

where $a,\, b,\, c \in \mathbb{C}$ are arbitrary scalars. Observing this we can then write

$$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

which says that

$$R = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

is a spanning set for UT_2 (Definition TSS). Is R is linearly independent? If so, it is a basis for UT_2. So consider a relation of linear dependence on R,

$$\alpha_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \mathcal{O} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

From this equation, one rapidly arrives at the conclusion that $\alpha_1 = \alpha_2 = \alpha_3 = 0$. So R is a linearly independent set (Definition LI), and hence is a basis (Definition B) for UT_2. Now, we simply count up the size of the set R to see that the dimension of UT_2 is $\dim(UT_2) = 3$.

5. A square matrix A of size n is upper-triangular if $[A]_{ij} = 0$ whenever $i > j$. Let UT_n be the set of all upper-triangular matrices of size n. Prove that UT_n is a subspace of the vector space of all square matrices of size n, M_{nn}. (15 points)

   Solution: Apply Theorem TSS.

   First, the zero vector of M_{nn} is the zero matrix, $\mathcal{O}$, whose entries are all zero (Definition ZM). This matrix then meets the condition that $[\mathcal{O}]_{ij} = 0$ for $i > j$ and so is an element of UT_n.

   Suppose $A, B \in UT_n$. Is $A + B \in UT_n$? We examine the entries of $A + B$ "below" the diagonal. That is, in the following, assume that $i > j$.

$$\begin{aligned} [A+B]_{ij} &= [A]_{ij} + [B]_{ij} && \text{Definition MA} \\ &= 0 + 0 && A, B \in UT_n \\ &= 0 \end{aligned}$$

   which qualifies $A + B$ for membership in UT_n.

   Suppose $\alpha \in \mathbb{C}$ and $A \in UT_n$. Is $\alpha A \in UT_n$? We examine the entries of αA "below" the diagonal. That is, in the following, assume that $i > j$.

$$\begin{aligned} [\alpha A]_{ij} &= \alpha\, [A]_{ij} && \text{Definition MSM} \\ &= \alpha 0 && A \in UT_n \\ &= 0 \end{aligned}$$

   which qualifies αA for membership in UT_n.

   Having fulfilled the three conditions of Theorem TSS we see that UT_n is a subspace of M_{nn}.

6. Suppose that V is a vector space. Then by Property AI we know that for every vector $\mathbf{v} \in V$, there is an additive inverse $-\mathbf{v} \in V$. Prove that the additive inverse is unique for each choice of $\mathbf{v}$. (15 points)

   Solution: This is Theorem AIU. A careful proof can be found there.