Math 232 Name: Key Dr. Beezer
Quiz D & E Spring 2005

Show *all* of your work and *explain* your answers fully. There is a total of 90 possible points.

1. Without using a calculator, compute the determinant of the matrix below. (15 points)

$$A = \begin{bmatrix} 2 & 0 & 3 & 2 \\ 5 & 1 & 2 & 4 \\ 3 & 0 & 1 & 2 \\ 5 & 3 & 2 & 1 \end{bmatrix}$$

Solution: With two zeros in column 2, we choose to expand about that column (Theorem DERC),

$$\begin{aligned}
\det(A) &= \begin{vmatrix} 2 & 0 & 3 & 2 \\ 5 & 1 & 2 & 4 \\ 3 & 0 & 1 & 2 \\ 5 & 3 & 2 & 1 \end{vmatrix} \\
&= 0(-1)\begin{vmatrix} 5 & 2 & 4 \\ 3 & 1 & 2 \\ 5 & 2 & 1 \end{vmatrix} + 1(1)\begin{vmatrix} 2 & 3 & 2 \\ 3 & 1 & 2 \\ 5 & 2 & 1 \end{vmatrix} + 0(-1)\begin{vmatrix} 2 & 3 & 2 \\ 5 & 2 & 4 \\ 5 & 2 & 1 \end{vmatrix} + 3(1)\begin{vmatrix} 2 & 3 & 2 \\ 5 & 2 & 4 \\ 3 & 1 & 2 \end{vmatrix} \\
&= (1)\left(2(1(1) - 2(2)) - 3(3(1) - 5(2)) + 2(3(2) - 5(1))\right) + \\
&\qquad (3)\left(2(2(2) - 4(1)) - 3(5(2) - 4(3)) + 2(5(1) - 3(2))\right) \\
&= (-6 + 21 + 2) + (3)(0 + 6 - 2) = 29
\end{aligned}$$

2. Without using a calculator, find the eigenvalues of the matrix below. (15 points)

$$B = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$$

Solution: The characteristic polynomial (Theorem CP) is

$$\begin{aligned}
p_A(x) &= \det(A - xI_2) \\
&= \begin{vmatrix} 2-x & -1 \\ 1 & 1-x \end{vmatrix} \\
&= (2-x)(1-x) - (1)(-1) && \text{Theorem DMST} \\
&= x^2 - 3x + 3 \\
&= \left(x - \frac{3+3i}{2}\right)\left(x - \frac{3-3i}{2}\right)
\end{aligned}$$

where the factorization can be obtained by finding the roots of $p_A(x) = 0$ with the quadratic equation. By Theorem EMRCP the eigenvalues of A are the complex numbers $\lambda_1 = \frac{3+3i}{2}$ and $\lambda_2 = \frac{3-3i}{2}$.

3. Determine if the matrix A below is diagonalizable. If the matrix is diagonalizable, then find a diagonal matrix D that is similar to A, and provide the invertible matrix S that perfoms the similarity transformation. You may use your calculator to find eigenvalues, but you may only use the row-reducing function of your calculator to assist with finding eigenvectors. (30 points)

$$A = \begin{bmatrix} 1 & 9 & 9 & 24 \\ -3 & -27 & -29 & -68 \\ 1 & 11 & 13 & 26 \\ 1 & 7 & 7 & 18 \end{bmatrix}$$

Solution: A calculator will provide the eigenvalues $\lambda = 2,\ 2,\ 1,\ 0$, so we can reconstruct the characteristic polynomial as

$$p_A(x) = (x-2)^2(x-1)x$$

so the algebraic multiplicities of the eigenvalues are

$$\alpha_A(2) = 2 \qquad \alpha_A(1) = 1 \qquad \alpha_A(0) = 1$$

Now compute eigenspaces by hand, obtaining null spaces for each of the three eigenvalues by constructing the correct singular matrix (Theorem EMNS),

$$A - 2I_4 = \begin{bmatrix} -1 & 9 & 9 & 24 \\ -3 & -29 & -29 & -68 \\ 1 & 11 & 11 & 26 \\ 1 & 7 & 7 & 16 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & -\frac{3}{2} \\ 0 & 1 & 1 & \frac{5}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$E_A(2) = \mathcal{N}(A - 2I_4) = \mathcal{S}p\left(\left\{\begin{bmatrix} \frac{3}{2} \\ -\frac{5}{2} \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}\right\}\right) = \mathcal{S}p\left(\left\{\begin{bmatrix} 3 \\ -5 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}\right\}\right)$$

$$A - 1I_4 = \begin{bmatrix} 0 & 9 & 9 & 24 \\ -3 & -28 & -29 & -68 \\ 1 & 11 & 12 & 26 \\ 1 & 7 & 7 & 17 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & -\frac{5}{3} \\ 0 & 1 & 0 & \frac{13}{3} \\ 0 & 0 & 1 & -\frac{5}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$E_A(1) = \mathcal{N}(A - I_4) = \mathcal{S}p\left(\left\{\begin{bmatrix} \frac{5}{3} \\ -\frac{13}{3} \\ \frac{5}{3} \\ 1 \end{bmatrix}\right\}\right) = \mathcal{S}p\left(\left\{\begin{bmatrix} 5 \\ -13 \\ 5 \\ 3 \end{bmatrix}\right\}\right)$$

$$A - 0I_4 = \begin{bmatrix} 1 & 9 & 9 & 24 \\ -3 & -27 & -29 & -68 \\ 1 & 11 & 13 & 26 \\ 1 & 7 & 7 & 18 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$E_A(0) = \mathcal{N}(A - I_4) = \mathcal{S}p\left(\left\{\begin{bmatrix} 3 \\ -5 \\ 2 \\ 1 \end{bmatrix}\right\}\right)$$

From this we can compute the dimensions of the eigenspaces to obtain the geometric multiplicities,

$$\gamma_A(2) = 2 \qquad \gamma_A(1) = 1 \qquad \gamma_A(0) = 1$$

For each eigenvalue, the algebraic and geometric multiplicities are equal and so by Theorem DMLE we now know that A is diagonalizable. The construction in Theorem DC suggests we form a matrix whose columns are eigenvectors of A

$$S = \begin{bmatrix} 3 & 0 & 5 & 3 \\ -5 & -1 & -13 & -5 \\ 0 & 1 & 5 & 2 \\ 2 & 0 & 3 & 1 \end{bmatrix}$$

Since $\det(S) = -1 \neq 0$, we know that S is nonsingular (Theorem SMZD), so the columns of S are a set of 4 linearly independent eigenvectors of A. By the proof of Theorem SMZD we know

$$S^{-1}AS = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

a diagonal matrix with the eigenvalues of A along the diagonal, in the same order as the associated eigenvectors appear as columns of S.

4. Suppose that B is a nonsingular matrix. Prove that AB is similar to BA. (15 points)

Solution: The nonsingular (invertible) matrix B will provide the desired similarity transformation,

$$\begin{aligned} B^{-1}(BA)B &= \left(B^{-1}B\right)(AB) && \text{Theorem MMA} \\ &= I_n AB && \text{Definition MI} \\ &= AB && \text{Theorem MMIM} \end{aligned}$$

5. Suppose that A is a square matrix. Prove that a single vector may not be an eigenvector of A for two different eigenvalues. (15 points)

Solution: Suppose that the vector $\mathbf{x} \neq \mathbf{0}$ is an eigenvector of A for the two eigenvalues λ and ρ, where $\lambda \neq \rho$. Then $\lambda - \rho \neq 0$, so

$$\begin{aligned} \mathbf{0} &\neq (\lambda - \rho)\mathbf{x} && \text{Theorem SMEZV} \\ &= \lambda\mathbf{x} - \rho\mathbf{x} && \text{Property DSAC} \\ &= A\mathbf{x} - A\mathbf{x} && \lambda,\ \rho \text{ eigenvalues of } A \\ &= \mathbf{0} && \text{Property AIC} \end{aligned}$$

which is a contradiction.