Math 290 Name: Key Dr. Beezer
Quiz V Fall 2006

Show *all* of your work and *explain* your answers fully. There is a total of 100 possible points. If you use a calculator or Mathematica on a problem be sure to write down both the input and output.

1. Let $\mathbf{y} = \begin{bmatrix} 8 \\ 2 \\ 1 \end{bmatrix}$ and let $S = \left\{ \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix} \right\}$. Is $\mathbf{y} \in \langle S \rangle$? (15 points)

2. Write the solutions to the system of equations below in vector form. (15 points)

$$\begin{aligned} -x_1 + x_3 + x_4 &= 1 \\ 3x_1 + 1x_2 - 2x_3 &= 0 \\ 4x_1 + 2x_2 - 2x_3 + 2x_4 &= 2 \end{aligned}$$

3. Let $A$ be the matrix below. (20 points)

$$\begin{bmatrix} 2 & 3 & 1 & 4 \\ 1 & 2 & 1 & 3 \\ -1 & 0 & 1 & 1 \end{bmatrix}$$

(a) Find a set $S$ so that $\mathcal{N}(A) = \langle S \rangle$.

Solution: Theorem SSNS provides formulas for a set $S$ with this property, but first we must row-reduce $A$

$$A \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & -1 & -1 \\ 0 & \boxed{1} & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_3$ and $x_4$ would be the free variables in the homogeneous system $\mathcal{LS}(A, \mathbf{0})$ and Theorem SSNS provides the set $S = \{\mathbf{z}_1, \mathbf{z}_2\}$ where

$$\mathbf{z}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \qquad \mathbf{z}_2 = \begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

(b) If $\mathbf{z} = \begin{bmatrix} 3 \\ -5 \\ 1 \\ 2 \end{bmatrix}$, then show that $\mathbf{z} \in \mathcal{N}(A)$.

Solution: Simply employ the components of the vector $\mathbf{z}$ as the variables in the homogeneous system $\mathcal{LS}(A, \mathbf{0})$. The three equations of this sytem evaluate as follows,

$$\begin{aligned} 2(3) + 3(-5) + 1(1) + 4(2) &= 0 \\ 1(3) + 2(-5) + 1(1) + 3(2) &= 0 \\ -1(3) + 0(-5) + 1(1) + 1(2) &= 0 \end{aligned}$$

Since each result is zero, $\mathbf{z}$ qualifies for membership in $\mathcal{N}(A)$.

(c) Write $\mathbf{z}$ as a linear combination of the vectors in $S$.

Solution: By Theorem SSNS we know this must be possible. Find scalars $\alpha_1$ and $\alpha_2$ so that

$$\alpha_1 \mathbf{z}_1 + \alpha_2 \mathbf{z}_2 = \alpha_1 \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \\ 1 \\ 2 \end{bmatrix} = \mathbf{z}$$

Theorem SLSLC allows us to convert this question into a question about a system of four equations in two variables. The augmented matrix of this system row-reduces to

$$\begin{bmatrix} \boxed{1} & 0 & 1 \\ 0 & \boxed{1} & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

A solution is $\alpha_1 = 1$ and $\alpha_2 = 2$. (Notice too that this solution is unique.)

4. Let $T$ be the set of vectors $T = \left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 6 \end{bmatrix} \right\}$. Find two proper subsets of $T$, named $R$ and $S$, and such that $\langle R\rangle = \langle T\rangle$ and $\langle S\rangle = \langle T\rangle$. Prove that one of these two sets ($R$ or $S$) is linearly independent.

Solution: Let $A$ be the matrix whose columns are the vectors in $T$. Then row-reduce $A$,

$$A \xrightarrow{\text{RREF}} B = \begin{bmatrix} \boxed{1} & 0 & 0 & 2 \\ 0 & \boxed{1} & 0 & -1 \\ 0 & 0 & \boxed{1} & 1 \end{bmatrix}$$

From Theorem BS we can form $R$ by choosing the columns of $A$ that correspond to the pivot columns of $B$. Theorem BS also gurantees that $R$ will be linearly independent.

$$R = \left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix} \right\}$$

That was easy. To find a second set will require a bit more work. From $B$ we can obtain a solution to $\mathcal{LS}(A, \mathbf{0})$, which by Theorem SLSLC will provide a nontrivial relation of linear dependence on the columns of $A$, which are the vectors in $T$. To wit, choose the free variable $x_4$ to be 1, then $x_1 = -2$, $x_2 = 1$, $x_3 = -1$, and so

$$(-2)\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + (1)\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + (-1)\begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix} + (1)\begin{bmatrix} 3 \\ 0 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

this equation can be rewritten with the second vector staying put, and the other three moving to the other side of the equality,

$$\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + = (2)\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + (1)\begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix} + (-1)\begin{bmatrix} 3 \\ 0 \\ 6 \end{bmatrix}$$

this is enough to conclude that the second vector in $T$ is "surplus" and can be replaced. So set

$$S = \left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 6 \end{bmatrix} \right\}$$

and then $\langle S\rangle = \langle T\rangle$. $T$ is also a linearly independent set, but we would need to show that directly.

5. Suppose that $S$ is a nonempty set of vectors from $\mathbb{C}^m$. Prove that $\mathbf{0} \in \langle S \rangle$ (i.e. the zero vector is an element of the span of $S$). (15 points)

   Solution: Let $S = \{\mathbf{v}_1,\, \mathbf{v}_2,\, \mathbf{v}_3,\, \ldots,\, \mathbf{v}_n\}$. Choose scalars, $\alpha_1 = 0,\, \alpha_2 = 0,\, \alpha_2 = 0,\, \ldots, \alpha_n = 0$. Then

$$\begin{aligned} \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3 + \cdots + \alpha_n\mathbf{v}_n &= 0\mathbf{v}_1 + 0\mathbf{v}_2 + 0\mathbf{v}_3 + 0\mathbf{v}_1 + \cdots + 0\mathbf{v}_n \\ &= \mathbf{0} + \mathbf{0} + \mathbf{0} + \cdots + \mathbf{0} && \text{Definition CVSM} \\ &= \mathbf{0} && \text{Definition CVA} \end{aligned}$$

   Since the zero vector, $\mathbf{0}$, is a linear combination of the vectors in $S$, we can say $\mathbf{0} \in \langle S \rangle$ by Definition SS.

6. Suppose that $\{\mathbf{v}_1,\, \mathbf{v}_2,\, \mathbf{v}_3,\, \ldots,\, \mathbf{v}_n\}$ is a linearly independent set of vectors.
   Prove that $\{\mathbf{v}_1 - \mathbf{v}_2,\, \mathbf{v}_2 - \mathbf{v}_3,\, \mathbf{v}_3 - \mathbf{v}_4,\, \ldots,\, \mathbf{v}_n - \mathbf{v}_1\}$ is a linearly dependent set. (15 points)

   Solution: Consider the following linear combination

$$\begin{aligned} 1\,(\mathbf{v}_1 - \mathbf{v}_2) &+ 1\,(\mathbf{v}_2 - \mathbf{v}_3) + 1\,(\mathbf{v}_3 - \mathbf{v}_4) + \cdots + 1\,(\mathbf{v}_n - \mathbf{v}_1) \\ &= \mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_2 - \mathbf{v}_3 + \mathbf{v}_3 - \mathbf{v}_4 + \cdots + \mathbf{v}_n - \mathbf{v}_1 \\ &= \mathbf{v}_1 + \mathbf{0} + \mathbf{0} + \cdots + \mathbf{0} - \mathbf{v}_1 \\ &= \mathbf{0} \end{aligned}$$