Math 290 Name: Key Dr. Beezer
Quiz M Fall 2006

Show *all* of your work and *explain* your answers fully. There is a total of 100 possible points. If you use a calculator or Mathematica on a problem be sure to write down both the input and output.

1. Use a matrix inverse to solve the linear system of equations below. Show the inverse and explain its application. No credit will be given for solutions obtained by other methods. (15 points)

$$\begin{aligned} x_1 - x_2 + 2x_3 &= 5 \\ x_1 - 2x_3 &= -8 \\ 2x_1 - x_2 - x_3 &= -6 \end{aligned}$$

Solution: We can reformulate the linear system as a vector equality with a matrix-vector product via Theorem SLEMM. The system is then represented by $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & -2 \\ 2 & -1 & -1 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 5 \\ -8 \\ -6 \end{bmatrix}$$

According to Theorem SNSCM, if A is nonsingular then the (unique) solution will be given by $A^{-1}\mathbf{b}$. We attempt the computation of A^{-1} through Theorem CINSM, or with our favorite computational device and obtain,

$$A^{-1} = \begin{bmatrix} 2 & 3 & -2 \\ 3 & 5 & -4 \\ 1 & 1 & -1 \end{bmatrix}$$

So by Theorem NSI, we know A is nonsingular, and so the unique solution is

$$A^{-1}\mathbf{b} = \begin{bmatrix} 2 & 3 & -2 \\ 3 & 5 & -4 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 5 \\ -8 \\ -6 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 3 \end{bmatrix}$$

2. For the 3×4 matrix A and the column vector $\mathbf{y} \in \mathbb{C}^4$ given below, determine if $\mathbf{y}$ is in the row space of A. In other words, answer the question: $\mathbf{y} \in \mathcal{R}(A)$? (15 points)

$$A = \begin{bmatrix} -2 & 6 & 7 & -1 \\ 7 & -3 & 0 & -3 \\ 8 & 0 & 7 & 6 \end{bmatrix} \qquad \mathbf{y} = \begin{bmatrix} 2 \\ 1 \\ 3 \\ -2 \end{bmatrix}$$

Solution:

$$\begin{aligned} \mathbf{y} \in \mathcal{R}(A) &\iff \mathbf{y} \in \mathcal{C}\left(A^t\right) && \text{Definition RSM} \\ &\iff \mathcal{LS}\left(A^t,\, \mathbf{y}\right) \text{ is consistent} && \text{Theorem CSCS} \end{aligned}$$

The augmented matrix $\left[A^t \,|\, \mathbf{y}\right]$ row reduces to

$$\begin{bmatrix} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & \boxed{1} \end{bmatrix}$$

and with a leading 1 in the final column Theorem RCLS tells us the linear system is inconsistent and so $\mathbf{y} \notin \mathcal{R}(A)$.

3. For the matrix A below find two different linearly independent sets whose spans equal the column space of A, $\mathcal{C}(A)$, according to the directions in each part. (20 points)

$$A = \begin{bmatrix} 3 & 5 & 1 & -2 \\ 1 & 2 & 3 & 3 \\ -3 & -4 & 7 & 13 \end{bmatrix}$$

(a) Obtain a set whose elements are each columns of A.

Solution: By Theorem BCS we can row-reduce A, identify pivot columns with the set D, and "keep" those columns of A and we will have a set with the desired properties.

$$A \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & -13 & -19 \\ 0 & \boxed{1} & 8 & 11 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So we have the set of pivot columns $D = \{1, 2\}$ and we "keep" the first two columns of A,

$$\left\{ \begin{bmatrix} 3 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \\ -4 \end{bmatrix} \right\}$$

(b) Obtain a second set by a procedure that is substantially different from the procedure in part (a).

Solution: We can view the column space as the row space of the transpose (Theorem CSRST). We can get a basis of the row space of a matrix quickly by bringing the matrix to reduced row-echelon form and keeping the nonzero rows as column vectors (Theorem BRS). Here goes.

$$A^t \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & -2 \\ 0 & \boxed{1} & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Taking the nonzero rows and tilting them up as columns gives us

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \right\}$$

4. For the matrix D below use the extended echelon form to find the requested sets. No credit will be given for using other methods. (20 points)

$$D = \begin{bmatrix} -7 & -11 & -19 & -15 \\ 6 & 10 & 18 & 14 \\ 3 & 5 & 9 & 7 \\ -1 & -2 & -4 & -3 \end{bmatrix}$$

Solution: For both parts, we need the extended echelon form of the matrix.

$$\begin{bmatrix} -7 & -11 & -19 & -15 & 1 & 0 & 0 & 0 \\ 6 & 10 & 18 & 14 & 0 & 1 & 0 & 0 \\ 3 & 5 & 9 & 7 & 0 & 0 & 1 & 0 \\ -1 & -2 & -4 & -3 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & -2 & -1 & 0 & 0 & 2 & 5 \\ 0 & \boxed{1} & 3 & 2 & 0 & 0 & -1 & -3 \\ 0 & 0 & 0 & 0 & \boxed{1} & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} & -2 & 0 \end{bmatrix}$$

From this matrix we extract the last two rows, in the last four columns to form the matrix L,

$$L = \begin{bmatrix} \boxed{1} & 0 & 3 & 2 \\ 0 & \boxed{1} & -2 & 0 \end{bmatrix}$$

(a) A linearly independent set whose span is the column space of D.

Solution: By Theorem FS and Theorem BNS we have

$$\mathcal{C}(D) = \mathcal{N}(L) = \left\langle \left\{ \begin{bmatrix} -3 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} \right\rangle$$

(b) A linearly independent set whose span is the left null space of D.

Solution: By Theorem FS and Theorem BRS we have

$$\mathcal{L}(D) = \mathcal{R}(L) = \left\langle \left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \\ 0 \end{bmatrix} \right\} \right\rangle$$

5. Suppose that $\alpha \in \mathbb{C}$ is a scalar, A is an $m \times n$ matrix and B is an $n \times p$ matrix. Prove that $\alpha(AB) = A(\alpha B)$. (Give a careful proof, do not just quote a result from the text.) (15 points)

   Solution: This part of Theorem MMSMM. Here's the proof.

$$\begin{aligned}
[\alpha(AB)]_{ij} &= \alpha\, [AB]_{ij} && \text{Definition MSM} \\
&= \alpha \sum_{k=1}^{n} [A]_{ik}\, [B]_{kj} && \text{Theorem EMP} \\
&= \sum_{k=1}^{n} \alpha\, [A]_{ik}\, [B]_{kj} && \text{Distributivity in } \mathbb{C} \\
&= \sum_{k=1}^{n} [A]_{ik}\, \alpha\, [B]_{kj} && \text{Commutativity in } \mathbb{C} \\
&= \sum_{k=1}^{n} [A]_{ik}\, [\alpha B]_{kj} && \text{Definition MSM} \\
&= [A(\alpha B)]_{ij} && \text{Theorem EMP}
\end{aligned}$$

   So the matrices $\alpha(AB)$ and $A(\alpha B)$ are equal, entry-by-entry, and by the definition of matrix equality (Definition ME) we can say they are equal matrices.

6. Suppose that $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$, $\mathbf{b} \in \mathbb{C}^m$ and A is an $m \times n$ matrix. If $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{x} + \mathbf{y}$ are each a solution to the linear system $\mathcal{LS}(A, \mathbf{b})$, what interesting can you say about $\mathbf{b}$? Form an implication with the existence of the three solutions as the hypothesis and an interesting statement about $\mathcal{LS}(A, \mathbf{b})$ as the conclusion, and then give a proof. (15 points)

   Solution: $\mathcal{LS}(A, \mathbf{b})$ must be homogeneous. To see this consider that

$$\begin{aligned}
\mathbf{b} &= A\mathbf{x} && \text{Theorem SLEMM} \\
&= A\mathbf{x} + \mathbf{0} && \text{Property ZC} \\
&= A\mathbf{x} + A\mathbf{y} - A\mathbf{y} && \text{Property AIC} \\
&= A\left(\mathbf{x} + \mathbf{y}\right) - A\mathbf{y} && \text{Theorem MMDAA} \\
&= \mathbf{b} - \mathbf{b} && \text{Theorem SLEMM} \\
&= \mathbf{0} && \text{Property AIC}
\end{aligned}$$

   By Definition HS we see that $\mathcal{LS}(A, \mathbf{b})$ is homogeneous.