Math 290 Name: Key Dr. Beezer
Quiz VS Fall 2006

Show *all* of your work and *explain* your answers fully. There is a total of 100 possible points. If you use a calculator or Mathematica on a problem be sure to write down both the input and output.

1. For the matrix A below, determine the dimensions of the null space, column space, row space and left null space. (15 points)

$$A = \begin{bmatrix} 1 & -1 & 5 & 2 & 9 \\ -2 & 2 & -10 & 1 & -8 \\ 1 & 1 & 1 & 1 & 9 \\ 3 & -1 & 11 & 0 & 17 \end{bmatrix}$$

Solution: Row-reduce A to reduced row-echelon form,

$$\begin{bmatrix} \boxed{1} & 0 & 3 & 0 & 6 \\ 0 & \boxed{1} & -2 & 0 & 1 \\ 0 & 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

to discover three nonzero rows. So $r = 3$ along with $m = 4$ rows and $n = 5$ columns. Then by Theorem DFS, $\dim(\mathcal{N}(A)) = n - r = 2$, $\dim(\mathcal{C}(A)) = r = 3$, $\dim(\mathcal{R}(A)) = r = 3$, $\dim(A) = m - r = 1$.

2. In the vector space M_{22}, determine if the set T is linearly independent. (15 points)

$$T = \left\{ \begin{bmatrix} -2 & 1 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 4 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 4 & -2 \\ 0 & 2 \end{bmatrix} \right\}$$

Solution: We begin with a relation of linear dependence on the set, employing the 2×2 zero matrix as the zero vector of M_{22},

$$a_1 \begin{bmatrix} -2 & 1 \\ -1 & 1 \end{bmatrix} + a_2 \begin{bmatrix} 2 & 2 \\ 2 & 1 \end{bmatrix} + a_3 \begin{bmatrix} -1 & 4 \\ 3 & 4 \end{bmatrix} + a_4 \begin{bmatrix} 4 & -2 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Using the definitions of scalar multiplication, vector addition and matrix equality from M_{22} (Example VSM), we arrive at a homogeneous system of four equations in the four variables a_1, a_2, a_3, a_4 with coefficient matrix,

$$\begin{bmatrix} -2 & 2 & -1 & 4 \\ 1 & 2 & 4 & -2 \\ -1 & 2 & 3 & 0 \\ 1 & 1 & 4 & 2 \end{bmatrix}$$

This coefficent matrix row-reduces to the identity matrix, so by Theorem NRRI the matrix is nonsingular, and the only solution for the four scalars is the trivial solution $a_1 = a_2 = a_3 = a_4 = 0$. By Definition LI, the set T is linearly independent.

3. The set W below is a subspace of P_3 (the vector space of polynomials of degree three or less) and $\dim(W) = 3$. You may assume these two facts throughout this problem. (40 points)

$$W = \{ p(x) \in P_3 \mid p(1) = 0 \}$$

(a) Prove that the set $B = \left\{2x - 2,\, x^2 - 1,\, x^3 - x^2 + x - 1\right\} \subset W$ is a basis for W.

Solution: Since $\dim(W) = 3$, Theorem G tells us that we need only determine if B is linearly independent, or B spans W, and then we get the "other" property for free. We'll check linear independence by starting with a relation of linear dependence.

$$b_1(2x-2) + b_2\left(x^2-1\right) + b_3\left(x^3 - x^2 + x - 1\right) = 0 + 0x + 0x^2 + 0x^3$$

Using vector addition, scalar multiplication and equality from Example VSP we arrive at a homogeneous system of four equations in the three variables b_1, b_2, b_3 having coefficient matrix

$$\begin{bmatrix} -2 & -1 & -1 \\ 2 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

which row-reduces to

$$\begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \end{bmatrix}$$

So we recognize there is just one solution (Theorem FVCS) of the consistent system (Theorem HSC), namely the trivial solution $b_1 = b_2 = b_3 = 0$. So B is linearly independent by Definition LI and by Theorem G is a basis of W.

(b) Is the set $C = \left\{5x - 5,\, x^2 - 2x + 1,\, x^2 - x,\, x^3 - x\right\}$ linearly independent? Why or why not?

Solution: You must first check that C is a subset of W before applying Theorem G. Each vector in C is a polynomial with a root at 1, so yes, $C \subset W$. Now Goldilocks tells us the set must be linearly dependent.

(c) Does the set $D = \left\{x^2 + 6x - 7,\, 4x^3 + 2x - 6\right\}$ span W? Why or why not?

Solution: As in part (b), check that $D \subset W$. Then Theorem G says we do not have enough vectors to span W.

(d) Find a basis for P_3 that has the set B as a subset.

Solution: The set B is linearly independent, since it is a basis of W. Theorem ELIS says we can choose a vector not in $W = \langle B\rangle$ and extend the linearly independent set. The polynomial $r(x) = 37$ does not have 1 as a root, so $r(x) \notin W$. (There are many, many possible choices for $r(x)$.)

The set $E = B \cup \{r(x)\}$ is then linearly independent in P_3. By Theorem DP, $\dim(P_3) = 3 + 1 = 4$. So we can apply Theorem G to conclude that B spans W. Finally, by Definition B, we see that E is a basis of P_3, and obviously has B as a subset.

4. Theorem TSS tells us that we can check three conditions to determine if a subset of a vector space is also a subspace. Illustrate the application of this theorem by showing that $W = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \,\middle|\, 3a - 2b + 6c = 0 \right\}$ is a subspace of $\mathbb{C}^3$. (15 points)

   Solution: This is entirely similar in style to Example SC3 and Exercise S.M20.

5. Suppose that $\alpha \in \mathbb{C}$ is any scalar. Using only the ten defining properties of a vector space, show that $\alpha\mathbf{0} = \mathbf{0}$. (15 points)

   Solution: This is Theorem ZVSM. See the proof given there.