Math 290 Name: Key Dr. Beezer
Quiz D&E Fall 2006

Show *all* of your work and *explain* your answers fully. There is a total of 100 possible points. If you use a calculator or Mathematica on a problem be sure to write down both the input and output.

1. Compute the determinant of the matrix A below by first using row operations to convert it to an upper-triangular matrix. (15 points)

$$A = \begin{bmatrix} 1 & 1 & -6 \\ 3 & 1 & 2 \\ -2 & -1 & 9 \end{bmatrix}$$

Solution:

$$\begin{vmatrix} 1 & 1 & -6 \\ 3 & 1 & 2 \\ -2 & -1 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 1 & -6 \\ 0 & -2 & 20 \\ -2 & -1 & 9 \end{vmatrix} \qquad \text{Theorem DRCMA,} \quad -3R_1 + R_2$$

$$= \begin{vmatrix} 1 & 1 & -6 \\ 0 & -2 & 20 \\ 0 & 1 & -3 \end{vmatrix} \qquad \text{Theorem DRCMA,} \quad 2R_1 + R_3$$

$$= \begin{vmatrix} 1 & 1 & -6 \\ 0 & -2 & 20 \\ 0 & 1 & -3 \end{vmatrix} \qquad \text{Theorem DRCMA,} \quad 2R_1 + R_3$$

$$= -\begin{vmatrix} 1 & 1 & -6 \\ 0 & 1 & -3 \\ 0 & -2 & 20 \end{vmatrix} \qquad \text{Theorem DRCS,} \quad R_2 \leftrightarrow R_3$$

$$= -\begin{vmatrix} 1 & 1 & -6 \\ 0 & 1 & -3 \\ 0 & 0 & 14 \end{vmatrix} \qquad \text{Theorem DRCMA,} \quad 2R_2 + R_3$$

$$= -(1)(1)(14) = -14$$

2. For the matrix B below, find the eigenvalues, eigenspaces, algebraic and geometric multiplicities, without the aid of a calculator. (20 points)

$$B = \begin{vmatrix} 7 & -9 & -6 \\ 6 & -8 & -6 \\ -2 & 3 & 3 \end{vmatrix}$$

Solution:

$$\begin{aligned}
p_B(x) &= \det(B - xI_3) && \text{Definition CP} \\
&= \begin{vmatrix} 7-x & -9 & -6 \\ 6 & -8-x & -6 \\ -2 & 3 & 3-x \end{vmatrix} \\
&= (7-x)\begin{vmatrix} -8-x & -6 \\ 3 & 3-x \end{vmatrix} - (-9)\begin{vmatrix} 6 & -6 \\ -2 & 3-x \end{vmatrix} + (-6)\begin{vmatrix} 6 & -8-x \\ -2 & 3 \end{vmatrix} && \text{Definition DM} \\
&= (7-x)(x^2+5x-6) + 9(6-6x) + (-6)(-2x+2) && \text{Theorem DMST} \\
&= -x^3 + 2x^2 - x \\
&= -x(x-1)^2
\end{aligned}$$

Roots of the characteristic polynomial are the eigenvalues of the matrix (Theorem EMRCP) and their algebraic multiplicities are the multiplicities of the roots (Definition AME),

$$\lambda = 0 \qquad \alpha_A(0) = 1 \qquad \lambda = 1 \qquad \alpha_A(1) = 2$$

Eigenspaces are null spaces (Definition EM) and geometric multiplicities are dimensions of these subspaces (Definition GME),

$\lambda = 0$

$$B - 0I_3 = \begin{bmatrix} 7 & -9 & -6 \\ 6 & -8 & -6 \\ -2 & 3 & 3 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$E_B(0) = \mathcal{N}(B - 0I_3) = \left\langle \left\{ \begin{bmatrix} -3 \\ -3 \\ 1 \end{bmatrix} \right\} \right\rangle$$

$\lambda = 1$

$$B - 1I_3 = \begin{bmatrix} 6 & -9 & -6 \\ 6 & -9 & -6 \\ -2 & 3 & 2 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & -\frac{3}{2} & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$E_B(1) = \mathcal{N}(B - 1I_3) = \left\langle \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} \right\} \right\rangle$$

$$\gamma_B(0) = 1 \qquad \gamma_B(1) = 2$$

3. Given the matrix C below, find a diagonal matrix D and a nonsingular matrix S such that $S^{-1}CS = D$. (You may use your calculator on this problem.) (20 points)

$$C = \begin{bmatrix} -5 & 4 & 4 & 0 \\ -8 & 7 & 8 & 0 \\ -3 & -3 & 2 & -6 \\ 6 & -6 & -6 & -1 \end{bmatrix}$$

Solution: With our favorite computing device we find eigenvalues and eigenspaces,

$$\lambda = 3 \qquad E_C(3) = \left\langle \left\{ \begin{bmatrix} -2 \\ -4 \\ 0 \\ 3 \end{bmatrix} \right\} \right\rangle$$

$$\lambda = 2 \qquad E_C(2) = \left\langle \left\{ \begin{bmatrix} -4 \\ -8 \\ 1 \\ 6 \end{bmatrix} \right\} \right\rangle$$

$$\lambda = -1 \qquad E_C(-1) = \left\langle \left\{ \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\} \right\rangle$$

We can choose these four basis vectors from the eigenspaces to be four eigenvectors that we hope form a linearly independent set. Create the matrix S by making these four vectors the columns,

$$S = \begin{bmatrix} -2 & -4 & -1 & 1 \\ -4 & -8 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 3 & 6 & 1 & 0 \end{bmatrix}$$

The matrix S row-reduces to the identity matrix, so we know S is nonsingular (Theorem NRRI), and the four eigenvectors are linearly independent. For each of these eigenvectors of C that is a column of S we can place the corresponding eigenvalue in the same column of a diagonal matrix D,

$$D = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Then the proof of Theorem DC says that $S^{-1}CS = D$, but you can go ahead and check the arithmetic if you don't believe it.

4. Suppose that λ is an eigenvalue of the $n \times n$ matrix A. Prove that $\lambda + \alpha$ is an eigenvalue of the matrix $A + \alpha I_n$. Do this "from scratch" (i.e. relying on definitions) and not by simply applying just one theorem from the book. (15 points)

   Solution: Suppose that $\mathbf{x}$ is an eigenvector of A for the eigenvalue λ. We will show that $\mathbf{x}$ is an eigenvector of $A + \alpha I_n$ for the eigenvalue $\lambda + \alpha$. To wit,

$$
\begin{aligned}
(A + \alpha I_n)\,\mathbf{x} &= A\mathbf{x} + \alpha I_n \mathbf{x} && \text{Theorem MMDAA}\\
&= \lambda\mathbf{x} + \alpha I_n \mathbf{x} && \text{Definition EEM}\\
&= \lambda\mathbf{x} + \alpha\mathbf{x} && \text{Theorem MMIM}\\
&= (\lambda + \alpha)\,\mathbf{x} && \text{Property DSAC}
\end{aligned}
$$

   So, by Definition EEM, $A + \alpha I_n$ has $\lambda + \alpha$ as an eigenvalue.

5. Suppose that A, B and C are square matrices of the same size, A is similar to B, and B is similar to C. Prove that A is similar to C. (15 points)

   Solution: This is part 3 of Theorem SER.

6. Suppose that H is a Hermitian matrix and S is an orthogonal matrix. Prove that $S^{-1}HS$ is a Hermitian matrix. (15 points)

   Solution: We appeal to the definition of a Hermitian matrix and compute,

$$
\begin{aligned}
\left(\overline{S^{-1}HS}\right)^t &= \overline{S^{-1}}\,\overline{H}\,\overline{S}^t && \text{Theorem MMCC}\\
&= \overline{S}^t\,\overline{H}^t\,\overline{S^{-1}}^t && \text{Theorem MMT}\\
&= \overline{S}^t H \overline{S^{-1}}^t && \text{Definition HM}\\
&= S^{-1}HS && \text{Theorem OMI}
\end{aligned}
$$

   So by Definition HM, the matrix $S^{-1}HS$ is Hermitian.