Math 290 | Name: Key | Dr. Beezer
Quiz LT | | Fall 2006

Show *all* of your work and *explain* your answers fully. There is a total of 100 possible points. If you use a calculator or Mathematica on a problem be sure to write down both the input and output.

1. Consider the linear transformation $T\colon P_2 \mapsto M_{22}$ defined below. (40 points)

$$T\left(a + bx + cx^2\right) = \begin{bmatrix} a + 2b + 3c & -a + b \\ 3a + b + 4c & 2a + 2c \end{bmatrix}$$

(a) Find the kernel of T, $\mathcal{K}(T)$.

Solution: The requirement that $a + bx + cx^2 \in \mathcal{K}(T)$ leads to the homogeneous system of equations,

$$\begin{aligned} a + 2b + 3c &= 0 \\ -a + b &= 0 \\ 3a + b + 4c &= 0 \\ 2a + 2c &= 0 \end{aligned}$$

with a coefficient matrix that row-reduces to

$$\begin{bmatrix} \boxed{1} & 0 & 1 \\ 0 & \boxed{1} & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So $a = -c$ and $b = -c$ and we can write

$$\mathcal{K}(T) = \left\{ (-c) + (-c)x + (c)x^2 \,\middle|\, c \in \mathbb{C} \right\} = \left\langle \left\{ -1 - x + x^2 \right\} \right\rangle$$

(b) Find the range of T, $\mathcal{R}(T)$.

Solution: We can massage a typical output of T as follows,

$$\begin{aligned} T\left(a + bx + cx^2\right) &= \begin{bmatrix} a + 2b + 3c & -a + b \\ 3a + b + 4c & 2a + 2c \end{bmatrix} \\ &= \begin{bmatrix} a & -a \\ 3a & 2a \end{bmatrix} + \begin{bmatrix} 2b & b \\ b & 0 \end{bmatrix} + \begin{bmatrix} 3c & 0 \\ 4c & 2c \end{bmatrix} \\ &= a \begin{bmatrix} 1 & -1 \\ 3 & 2 \end{bmatrix} + b \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} + c \begin{bmatrix} 3 & 0 \\ 4 & 2 \end{bmatrix} \end{aligned}$$

This says that

$$\mathcal{R}(T) = \left\langle \left\{ \begin{bmatrix} 1 & -1 \\ 3 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 3 & 0 \\ 4 & 2 \end{bmatrix} \right\} \right\rangle$$

We can improve on this expression for the range, since this spanning set is linearly dependent. An arbitrary relation of linear dependence on this set leads to a system of equations having $a_1 = 1$, $a_2 = 1$, and $a_3 = -1$ as one nontrivial solution. This would allow us to remove any one of these three matrices and then arrive at a basis with two vectors (we can determine in advance that the range has dimension 2 with Theorem RPNDD).

Another approach on this problem is to apply Theorem SSRLT and evaluate T on a nice basis, such as $\{1,\, x,\, x^2\}$, to get a spanning set for $\mathcal{R}(T)$. For this particular choice of basis, this quickly leads to the exact same spanning set of three matrices.

(c) If possible, find two different elements of P_2, $p(x)$, $q(x)$, so that $T\left(p(x)\right) = T\left(q(x)\right)$.

Solution: Set $p(x)$ to any polynomial, say $p(x) = 4x + 3x^2$. then choose a member of the kernel of T, say $z(x) = -2 - 2x + 2x^2 \in \mathcal{K}(T)$. Set $q(x) = z(x) + p(x) = -2 + 2x + 5x^2$. In this case, verify that

$$T\left(p(x)\right) = \begin{bmatrix} 17 & 4 \\ 16 & 6 \end{bmatrix} = T\left(q(x)\right)$$

(d) If possible, find an element of M_{22}, A, so that there is no $p(x) \in P(x)$ with $T\left(p(x)\right) = A$.

Solution: The desired element will be any that lies outside of the range of T, so we want $A \notin \mathcal{R}(T)$. Suppose the first two elements of the spanning set in the answer to part (b) are chosen as a basis for the range. If we take one of these elements and "adjust" it in just one entry, so that we are not making an adjustment that is equal to adding a scalar multiple of the second basis vector, then such a matrix should lie outside the range. To wit, set

$$A = \begin{bmatrix} 1 & 15 \\ 3 & 2 \end{bmatrix}$$

The system of equations that results from setting $T\left(p(x)\right) = A$ is inconsistent, establishing that A has the desired property.

2. Prove that the linear transformation $S\colon \mathbb{C}^3 \mapsto \mathbb{C}^3$ below is invertible. (15 points)

$$S\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} -2a+b \\ -a+5b+4c \\ 4a+b-c \end{bmatrix}$$

Solution: The input vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is an element of the kernel, $\mathcal{K}(S)$, only if it is a solution to the homogeneous system,

$$\begin{aligned} -2a+b&=0 \\ -a+5b+4c&=0 \\ 4a+b-c&=0 \end{aligned}$$

which you can verify has only the trivial solution $a=b=c=0$. So $\mathcal{K}(S)=\{\mathbf{0}\}$ and S has nullity $n\left(S\right)=0$. By Theorem KILT, we know S is injective.

From Theorem RPNDD we have $n\left(S\right)+r\left(S\right)=\dim\left(\mathbb{C}^3\right)=3$, which leads to the conclusion that the rank of S is $r\left(S\right)=3$. So the range of S is a subspace (Theorem RLTS) of dimension 3 in $\mathbb{C}^3$. Theorem EDYES implies that $\mathcal{R}(S)=\mathbb{C}^3$, so S is surjective (Theorem RSLT).

Since S is both injective and surjective, S is an invertible linear transformation (Theorem ILTIS).

3. The system of equations below is inconsistent (you may assume this). Express this fact using the language of linear transformations. (15 points)

$$\begin{aligned} 2x_1+x_2+3x_3+5x_4&=-4 \\ x_1+x_2+x_3+4x_4&=1 \\ 3x_1+2x_2+4x_3+9x_4&=-1 \end{aligned}$$

Solution: Consider the linear transformation $Q\colon \mathbb{C}^4 \mapsto \mathbb{C}^3$ defined as $Q\left(\mathbf{x}\right)=Bx$ where B is the 3×4 coefficient matrix of the linear system,

$$B=\begin{bmatrix} 2 & 1 & 3 & 5 \\ 1 & 1 & 1 & 4 \\ 3 & 2 & 4 & 9 \end{bmatrix}$$

Then the solution set to the given system is the preimage of $\begin{bmatrix} -4 \\ 1 \\ -1 \end{bmatrix}$ under the linear transformation Q. The statement that the system is inconsistent is equivalent to saying that the preimage is the empty set,

$$Q^{-1}\left(\begin{bmatrix} -4 \\ 1 \\ -1 \end{bmatrix}\right)=\emptyset$$

4. The set D below is a basis of $\mathbb{C}^3$ (you may assume this). The linear transformation $T\colon \mathbb{C}^3 \mapsto \mathbb{C}^3$ is defined by its values on this basis. Prove that T is invertible and give a definition of T^{-1} in the same style as the definition of T. (15 points)

$$D = \left\{ \begin{bmatrix}1\\6\\2\end{bmatrix},\ \begin{bmatrix}0\\5\\0\end{bmatrix},\ \begin{bmatrix}-1\\-1\\4\end{bmatrix} \right\}$$

$$T\left(\begin{bmatrix}1\\6\\2\end{bmatrix}\right) = \begin{bmatrix}5\\1\\-4\end{bmatrix} \qquad T\left(\begin{bmatrix}0\\5\\0\end{bmatrix}\right) = \begin{bmatrix}9\\8\\-2\end{bmatrix} \qquad T\left(\begin{bmatrix}-1\\-1\\4\end{bmatrix}\right) = \begin{bmatrix}5\\1\\7\end{bmatrix}$$

Solution: Let E denote the set of the three specific outputs of T. If we make these three vectors the columns of a 3×3 matrix, we can row-reduce to determine that the matrix is nonsingular (Theorem NMRRI). By Theorem CNMB we then know that E is a basis of $\mathbb{C}^3$. Since E is linearly independent (Definition B), Theorem ILTB tells us that T is injective. Since E spans $\mathbb{C}^3$ (Definition B), Theorem SLTB tells us that T is surjective. Now apply Theorem ILTIS to see that T is invertible.

From the proof of Theorem ILTIS we learn that we can define T^{-1} simply by "reversing" the direction of the function and using the basis E for the definition,

$$T^{-1}\left(\begin{bmatrix}5\\1\\-4\end{bmatrix}\right) = \begin{bmatrix}1\\6\\2\end{bmatrix} \qquad T^{-1}\left(\begin{bmatrix}9\\8\\-2\end{bmatrix}\right) = \begin{bmatrix}0\\5\\0\end{bmatrix} \qquad T^{-1}\left(\begin{bmatrix}5\\1\\7\end{bmatrix}\right) = \begin{bmatrix}-1\\-1\\4\end{bmatrix}$$

5. Prove that the function $R\colon \mathbb{C}^2 \mapsto P_2$ below is a linear transformation. (15 points)

$$R\left(\begin{bmatrix}a\\b\end{bmatrix}\right) = (3a - b) + (4b)x + (a + 2b)x^2$$

Solution: By Definition LT we must establish two general equalities for R,

$$\begin{aligned}
R\left(\begin{bmatrix}a\\b\end{bmatrix} + \begin{bmatrix}c\\d\end{bmatrix}\right) &= R\left(\begin{bmatrix}a+c\\b+d\end{bmatrix}\right) && \text{Definition CVA}\\
&= (3(a+c) - (b+d)) + (4(b+d))x + ((a+c) + 2(b+d))x^2\\
&= ((3a-b) + (3c-d)) + (4b+4d)x + ((a+2b) + (c+2d))x^2\\
&= \left((3a-b) + (4b)x + (a+2b)x^2\right) + \left((3c-d) + (4d)x + (c+2d)x^2\right) && \text{Example VSP}\\
&= R\left(\begin{bmatrix}a\\b\end{bmatrix}\right) + R\left(\begin{bmatrix}c\\d\end{bmatrix}\right)
\end{aligned}$$

and

$$\begin{aligned}
R\left(\alpha\begin{bmatrix}a\\b\end{bmatrix}\right) &= R\left(\begin{bmatrix}\alpha a\\\alpha b\end{bmatrix}\right) && \text{Definition CVSM}\\
&= (3(\alpha a) - (\alpha b)) + (4(\alpha b))x + ((\alpha a) + 2(\alpha b))x^2\\
&= (\alpha(3a-b)) + (\alpha(4b))x + (\alpha(a+2b))x^2\\
&= \alpha\left((3a-b) + (4b)x + (a+2b)x^2\right) && \text{Example VSP}\\
&= \alpha R\left(\begin{bmatrix}a\\b\end{bmatrix}\right)
\end{aligned}$$