Math 290 Name: Key Dr. Beezer
Quiz R Fall 2006

Show *all* of your work and *explain* your answers fully. There is a total of 100 possible points. If you use a calculator or Mathematica on a problem be sure to write down both the input and output.

1. For the linear transformation $S\colon \mathbb{C}^2 \mapsto P_2$, compute the matrix representation relative to the bases B and C as given. In each case demonstrate the Fundamental Theorem of Matrix Representations by using the representation to compute $S\left(\begin{bmatrix}2\\-1\end{bmatrix}\right)$. (35 points)

$$S\left(\begin{bmatrix}a\\b\end{bmatrix}\right) = (-a+3b) + (2a+b)\,x + (a+3b)\,x^2$$

(a) $B = \left\{\begin{bmatrix}1\\0\end{bmatrix},\ \begin{bmatrix}0\\1\end{bmatrix}\right\}$, C=$\{1,\, x,\, x^2\}$

Solution: Apply S to each vector of B and form a vector representation relative to C,

$$\rho_C\left(S\left(\begin{bmatrix}1\\0\end{bmatrix}\right)\right) = \rho_C\left(-1+2x+x^2\right) = \begin{bmatrix}-1\\2\\1\end{bmatrix}$$

$$\rho_C\left(S\left(\begin{bmatrix}0\\1\end{bmatrix}\right)\right) = \rho_C\left(3+x+3x^2\right) = \begin{bmatrix}3\\1\\3\end{bmatrix}$$

Form a matrix with these vector representations as the columns, according to Definition MR,

$$M^S_{B,C} = \begin{bmatrix}-1 & 3\\ 2 & 1\\ 1 & 3\end{bmatrix}$$

Demonstrating Theorem FTMR, we see

$$\begin{aligned}
S\left(\begin{bmatrix}2\\-1\end{bmatrix}\right) &= \rho_C^{-1}\left(M^S_{B,C}\rho_B\left(\begin{bmatrix}2\\-1\end{bmatrix}\right)\right)\\
&= \rho_C^{-1}\left(M^S_{B,C}\begin{bmatrix}2\\-1\end{bmatrix}\right)\\
&= \rho_C^{-1}\left(\begin{bmatrix}-5\\3\\-1\end{bmatrix}\right)\\
&= -5+3x-x^2
\end{aligned}$$

(b) $B = \left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \end{bmatrix} \right\}$ C=$\left\{9 + 8x - 3x^2,\ 3 + 3x - x^2,\ -2 - 2x + x^2\right\}$

Solution: Apply S to each vector of B and form a vector representation relative to C,

$$\begin{aligned}
\rho_C\left(S\left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right)\right) &= \rho_C\left(7x + 6x^2\right) \\
&= \rho_C\left(-7(9 + 8x - 3x^2) + 33(3 + 3x - x^2) + 18(-2 - 2x + x^2)\right) \\
&= \begin{bmatrix} -7 \\ 33 \\ 18 \end{bmatrix} \\
\rho_C\left(S\left(\begin{bmatrix} 5 \\ 2 \end{bmatrix}\right)\right) &= \rho_C\left(1 + 12x + 11x^2\right) \\
&= \rho_C\left(-11(9 + 8x - 3x^2) + 56(3 + 3x - x^2) + 34(-2 - 2x + x^2)\right) \\
&= \begin{bmatrix} -11 \\ 56 \\ 34 \end{bmatrix}
\end{aligned}$$

Form a matrix with these vector representations as the columns, according to Definition MR,

$$M^S_{B,C} = \begin{bmatrix} -7 & -11 \\ 33 & 56 \\ 18 & 34 \end{bmatrix}$$

Demonstrating Theorem FTMR, we see

$$\begin{aligned}
S\left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right) &= \rho_C^{-1}\left(M^S_{B,C}\rho_B\left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right)\right) \\
&= \rho_C^{-1}\left(M^S_{B,C}\begin{bmatrix} 9 \\ -5 \end{bmatrix}\right) \\
&= \rho_C^{-1}\left(\begin{bmatrix} -8 \\ 17 \\ -8 \end{bmatrix}\right) \\
&= -8(9 + 8x - 3x^2) + 17(3 + 3x - x^2) + (-8)(-2 - 2x + x^2) \\
&= -5 + 3x - x^2
\end{aligned}$$

2. Consider the linear transformation $T\colon P_2 \mapsto M_{22}$ defined below. Find the kernel of T, $\mathcal{K}(T)$. (15 points)

$$T\left(a + bx + cx^2\right) = \begin{bmatrix} a + 2b + 3c & -a + b \\ 3a + b + 4c & 2a + 2c \end{bmatrix}$$

Solution: Begin with a matrix representation of T, relative to the nicest possible bases, say

$$B = \left\{1,\ x,\ x^2\right\} \qquad C = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

which yields

$$M^T_{B,C} = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 1 & 0 \\ 3 & 1 & 4 \\ 2 & 0 & 2 \end{bmatrix}$$

The kernel of T is isomorphic to the null space of $M^T_{B,C}$ via the "uncoordinatization" linear transformation ρ_B^{-1} (Theorem KNSI), so we first compute the null space of the matrix representation, row-reducing the matrix and using Theorem BNS,

$$\mathcal{N}\left(M^T_{B,C}\right) = \left\langle\left\{\begin{bmatrix}-1\\-1\\1\end{bmatrix}\right\}\right\rangle$$

Applying ρ_B^{-1} to the lone basis vector creates the polynomial $-1 - x + x^2$, which we can use as the basis vector for the kernel,

$$\mathcal{K}(T) = \left\langle\left\{-1 - x + x^2\right\}\right\rangle$$

3. Find a basis for M_{22} so that the linear transformation R below has a diagonal matrix representation. (20 points)

$$R\colon M_{22} \mapsto M_{22}, \quad R\left(\begin{bmatrix}a & b\\c & d\end{bmatrix}\right) = \begin{bmatrix}-17a + 12b - 36c + 6d & -6a + 4b - 12c\\ 7a - 5b + 15c - 3d & a + 2c + 2d\end{bmatrix}$$

Solution: Build a matrix representation of R. We can use any basis, so use the simplest possible, such as

$$B = \left\{\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right\}$$

Then the matrix representation is

$$A = M^R_{B,B} = \begin{bmatrix}-17 & 12 & -36 & 6\\-6 & 4 & -12 & 0\\7 & -5 & 15 & -3\\1 & 0 & 2 & 2\end{bmatrix}$$

and the eigenvalues of $M^R_{B,B}$ will be the eigenvalues of R. Similarly, we can extract the eigenvectors of R from the eigenvectors of $M^R_{B,B}$ (Theorem EER). Using techniques from Chapter E we find the eigenspaces,

$$\mathcal{E}_A(2) = \left\langle\left\{\begin{bmatrix}6\\0\\-3\\1\end{bmatrix}\right\}\right\rangle \qquad \mathcal{E}_A(1) = \left\langle\left\{\begin{bmatrix}-1\\-2\\0\\1\end{bmatrix}, \begin{bmatrix}-2\\0\\1\\0\end{bmatrix}\right\}\right\rangle \qquad \mathcal{E}_A(0) = \left\langle\left\{\begin{bmatrix}-6\\-3\\2\\1\end{bmatrix}\right\}\right\rangle$$

With algebraic multiplicities equal to geometric multiplicities, we can combine bases for each eigenspace to arrive at a basis for $\mathbb{C}^4$ (Theorem DMFE). In turn, we can "un-coordinatize" each of these eigenvectors for A to arrive at an eigenvector of R. The four basis vectors above become the basis

$$C = \left\{\begin{bmatrix}6 & 0\\-3 & 1\end{bmatrix}, \begin{bmatrix}-1 & -2\\0 & 1\end{bmatrix}, \begin{bmatrix}-2 & 0\\1 & 0\end{bmatrix}, \begin{bmatrix}-6 & -3\\2 & 1\end{bmatrix}\right\}$$

Though not requested, the resulting diagonal matrix representation of R, relative to C, is then

$$M^R_{C,C} = \begin{bmatrix}2 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 0\end{bmatrix}$$

4. A linear transformation $T\colon V \mapsto V$ is called **nilpotent** if there is a positive integer d such that $T^d(\mathbf{v}) = \mathbf{0}_V$ for all $\mathbf{v} \in V$. (The power d on T indicates repeated compositions.) Show that the linear transformation $T\colon P_2 \mapsto P_2$ below is nilpotent (where P_2 is the vector space of polynomials with degree at most 2). (15 points)

$$T(a + bx + c^2) = (-14a - 18b - 4c) + (9a + 12b + 2c)\,x + (7a + 9b + 2c)\,x^2$$

Solution: You might be inclined to repeatedly compose T with itself, even though this gets tedious and error-prone very quickly. Instead, build a matrix representation and replace composition by matrix multiplication (Theorem MRCLT). Relative to the standard basis $B = \left\{1,\, x,\, x^2\right\}$ we easily obtain the representation,

$$P = M^T_{B,B} = \begin{bmatrix} -14 & -18 & -4 \\ 9 & 12 & 2 \\ 7 & 9 & 2 \end{bmatrix}$$

Successive powers yield,

$$P^2 = \begin{bmatrix} 6 & 0 & 12 \\ -4 & 0 & -8 \\ -3 & 0 & -6 \end{bmatrix} \qquad P^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Since $P^3 = \mathcal{O}$, $T^3(p(x)) = 0 + 0x + 0x^2 = \mathbf{0}$ for all $p(x) \in P_2$, and we see that T is nilpotent.

5. Prove that the only eigenvalue of a nilpotent linear transformation is zero. (See the previous problem for the definition of a nilpotent linear transformation.) (15 points)

Solution: Let $\mathbf{x}$ be an eigenvector of a linear transformation T for the eigenvalue λ, and suppose that T is nilpotent with index p. Then

$$\begin{aligned}
\mathbf{0} &= T^p(\mathbf{x}) \\
&= T^{p-1}(T(\mathbf{x})) \\
&= T^{p-1}(\lambda\mathbf{x}) \\
&= \lambda T^{p-1}(\mathbf{x}) \\
&= \lambda T^{p-2}(T(\mathbf{x})) \\
&= \lambda T^{p-2}(\lambda\mathbf{x}) \\
&= \lambda^2 T^{p-2}(\mathbf{x}) \\
&\;\;\vdots \\
&= \lambda^p \mathbf{x}
\end{aligned}$$

Because $\mathbf{x}$ is an eigenvector, it is nonzero, and therefore Theorem SMEZV tells us that $\lambda^p = 0$ and so $\lambda = 0$.