Math 290 Name: Key Dr. Beezer
Quiz SLE Code: Fall 2007

Show *all* of your work and *explain* your answers fully. There is a total of 105 possible points. If you use a calculator or Mathematica on a problem be sure to write down both the input and output.

1. Find all solutions to the system of equations below. Express your solution(s) as a set of column vectors. (15 points)

$$\begin{aligned} 2x_1 + x_2 &= 6 \\ -x_1 - x_2 &= -2 \\ 3x_1 + 4x_2 &= 4 \\ 3x_1 + 5x_2 &= 2 \end{aligned}$$

Solution: We form the augmented matrix of the system,

$$\begin{bmatrix} 2 & 1 & 6 \\ -1 & -1 & -2 \\ 3 & 4 & 4 \\ 3 & 5 & 2 \end{bmatrix}$$

which row-reduces to

$$\begin{bmatrix} \boxed{1} & 0 & 4 \\ 0 & \boxed{1} & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

With no leading 1 in the final column, this system is consistent (Theorem RCLS). There are $n = 2$ variables in the system and $r = 2$ non-zero rows in the row-reduced matrix. By Theorem FVCS, there are $n-r = 2-2 = 0$ free variables and we therefore know the solution is unique. Forming the system of equations represented by the row-reduced matrix, we see that $x_1 = 4$ and $x_2 = -2$. Written as set of column vectors,

$$S = \left\{ \begin{bmatrix} 4 \\ -2 \end{bmatrix} \right\}$$

2. Find all solutions to the system of equations below. Express your solution(s) as a set of column vectors. (15 points)

$$\begin{aligned} -2x_1 + 2x_2 + x_3 - 3x_4 &= -10 \\ x_1 - x_2 \qquad + 2x_4 &= 3 \\ 3x_1 - 3x_2 + 4x_3 + 10x_4 &= -7 \end{aligned}$$

Solution: We form the augmented matrix of the system,

$$\begin{bmatrix} -2 & 2 & 1 & -3 & -10 \\ 1 & -1 & 0 & 2 & 3 \\ 3 & -3 & 4 & 10 & -7 \end{bmatrix}$$

which row-reduces to

$$\begin{bmatrix} \boxed{1} & -1 & 0 & 2 & 3 \\ 0 & 0 & \boxed{1} & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

With no leading 1 in the final column, this system is consistent (Theorem RCLS). There are $n = 4$ variables in the system and $r = 2$ non-zero rows in the row-reduced matrix. By Theorem FVCS, there are $n-r = 4-2 = 2$ free variables. More specifically, the free variables are indicated by all but the final index in $F = \{2, 4, 5\}$, namely x_2 and x_4. Rearranging each equation represented by a row of the row-reduced matrix, for the dependent variables with indices in $D = \{1, 3\}$, we find

$$x_1 = 3 + x_2 - 2x_4$$
$$x_3 = -4 - x_4$$

Written as set of column vectors,

$$S = \left\{ \left. \begin{bmatrix} 3 + x_2 - 2x_4 \\ x_2 \\ -4 - x_4 \\ x_4 \end{bmatrix} \right| x_2, x_4 \in \mathbb{C} \right\}$$

3. Is the matrix B below nonsingular? Explain clearly why or why not. (15 points)

$$B = \begin{bmatrix} 3 & 3 & -1 & 0 \\ 1 & 3 & 2 & 3 \\ 4 & 1 & 4 & 5 \\ 2 & 0 & 3 & 0 \end{bmatrix}$$

Solution: Row-reducing B yields,

$$\begin{bmatrix} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & \boxed{1} \end{bmatrix}$$

By Theorem NMRRI the matrix is nonsingular.

Notice that this solution makes no reference to any systems of equations, but uses the power of a theorem (whose proof does discuss systems of equations).

4. Find a matrix that is row-equivalent matrix to C and in reduced row-echelon form. Perform the necessary computations by hand, not with a calculator, and show your intermediate steps along with the row operations you've performed. (15 points)

$$C = \begin{bmatrix} 1 & -2 & -1 \\ 2 & -3 & 0 \\ -3 & 4 & -1 \end{bmatrix}$$

Solution: Following the algorithm of Theorem REMEF, and working to create pivot columns from left to right, we have

$$\begin{bmatrix} 1 & -2 & -1 \\ 2 & -3 & 0 \\ -3 & 4 & -1 \end{bmatrix} \xrightarrow{-2R_1+R_2} \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 2 \\ -3 & 4 & -1 \end{bmatrix} \xrightarrow{3R_1+R_3} \begin{bmatrix} \boxed{1} & -2 & -1 \\ 0 & 1 & 2 \\ 0 & -2 & -4 \end{bmatrix} \xrightarrow{2R_2+R_1}$$

$$\begin{bmatrix} \boxed{1} & 0 & 3 \\ 0 & 1 & 2 \\ 0 & -2 & -4 \end{bmatrix} \xrightarrow{1R_2+R_3} \begin{bmatrix} \boxed{1} & 0 & 3 \\ 0 & \boxed{1} & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

5. Find the null space of the matrix D below. (15 points)

$$D = \begin{bmatrix} 2 & 1 & 7 & -1 \\ -1 & 1 & 1 & 2 \\ -3 & 4 & 6 & 7 \end{bmatrix}$$

Solution: We form the augmented matrix of the homogeneous system $\mathcal{LS}(D, \mathbf{0})$ and row-reduce the matrix,

$$\begin{bmatrix} 2 & 1 & 7 & -1 & 0 \\ -1 & 1 & 1 & 2 & 0 \\ -3 & 4 & 6 & 7 & 0 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 2 & -1 & 0 \\ 0 & \boxed{1} & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We knew ahead of time that this system would be consistent (Theorem HSC), but we can now see there are $n-r = 4-2 = 2$ free variables, namely x_3 and x_4 (Theorem FVCS). Based on this analysis, we can rearrange the equations associated with each nonzero row of the reduced row-echelon form into an expression for the lone dependent variable as a function of the free variables. We arrive at the solution set to the homogeneous system, which is the null space of the matrix by Definition NSM,

$$\mathcal{N}(B) = \left\{ \left. \begin{bmatrix} -2x_3 + x_4 \\ -3x_3 - x_4 \\ x_3 \\ x_4 \end{bmatrix} \right| x_3,\, x_4 \in \mathbb{C} \right\}$$

6. Say **as much as possible** about solutions to the systems of linear equations described below. Give concrete reasons for your conclusions. (15 points)

(a) Consistent with 7 equations and 12 variables.

Solution: $m = 12 > 7 = n$, so Theorem CMVEI tells us there will be infinitely many solutions. Then, Theorem FVCS tells us there will be $n - r \geq n - m = 12 - 7 = 5$ free variables.

(b) Homogeneous.

Solution: The zero vector is a solution. There could be more. The existence of this solution implies the system is consistent, so this answer is better than just saying the system is consistent.

(c) 8 equations and 8 variables.

Solution: Anything could happen. In other words, this is not enough information to derive any more insight than what is provided by Theorem PSSLS. So we can just say there is no solution, one solution, or infinitely many solutions.

7. Consider the system of linear equations $\mathcal{LS}(A, \mathbf{b})$, and suppose that every element of a certain column of A is a common multiple of the corresponding element of the column vector $\mathbf{b}$. More precisely, there is a complex number α, and a column index j, such that $[A]_{ij} = \alpha\,[\mathbf{b}]_i$ for all i. Prove that the system is consistent. (15 points)

Solution: The condition about the multiple of the column translates to the following being a solution of the system $\mathcal{LS}(A, \mathbf{b})$,

$$x_1 = 0 \quad x_2 = 0 \quad \ldots \quad x_{j-1} = 0 \quad x_j = \alpha \quad x_{j+1} = 0 \quad \ldots \quad x_{n-1} = 0 \quad x_n = 0$$

With one solution of the system in hand, we can say the system is consistent (Definition CS).