Math 290 Name: Key Dr. Beezer
Quiz M Fall 2007

Show *all* of your work and *explain* your answers fully. There is a total of 110 possible points. If you use a calculator or Mathematica on a problem be sure to write down both the input and output.

1. Using only the "reduced row-echelon form" function of your calculator, demonstrate computing the inverse of A, A^{-1}. (10 points)

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 6 \end{bmatrix}$$

Solution: Apply Theorem CINM,

$$M = \begin{bmatrix} 1 & 1 & 2 & 1 & 0 & 0 \\ 2 & 1 & 3 & 0 & 1 & 0 \\ 3 & 2 & 6 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 0 & 0 & 2 & -1 \\ 0 & \boxed{1} & 0 & 3 & 0 & -1 \\ 0 & 0 & \boxed{1} & -1 & -1 & 1 \end{bmatrix} = N$$

We recognize from the first three columns of this computation that A is nonsingular (Theorem NMRRI), so Theorem CIMN and Theorem OSIS tell us the last three columns of N are the inverse of A,

$$A^{-1} = \begin{bmatrix} 0 & 2 & -1 \\ 3 & 0 & -1 \\ -1 & -1 & 1 \end{bmatrix}$$

2. Employing A^{-1} from the previous problem, find the solution set for $\mathcal{LS}(A, \mathbf{b})$ where $\mathbf{b} = \begin{bmatrix} 2 \\ -4 \\ 3 \end{bmatrix}$. Comment on theorems you could have used to predict the nature of the solution set beforehand. (10 points)

Solution: Theorem NMUS tells us that a system of equations with a nonsingular coefficient matrix will have a unique solution. Theorem SNCM, which we will now apply, tells us the solution is unique, but also gives an explicit expression for the solution,

$$\begin{aligned} \mathbf{x} &= A^{-1}\mathbf{b} \\ &= \begin{bmatrix} 0 & 2 & -1 \\ 3 & 0 & -1 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -4 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} -11 \\ 3 \\ 5 \end{bmatrix} \end{aligned}$$

3. Determine if the matrix B below is unitary or not. Explain the reasons for your answer. (10 points)

$$B = \begin{bmatrix} \frac{2-i}{\sqrt{30}} & \frac{15+5i}{\sqrt{300}} \\ \frac{4+3i}{\sqrt{30}} & \frac{-1-7i}{\sqrt{300}} \end{bmatrix}$$

Solution: Check that $B^*B = I_2$ (and hence B is unitary by Definition UM) most efficiently as follows. Let $\mathbf{x}$ be the first column of B and let $\mathbf{y}$ be the second column of B. Then

- $\langle \mathbf{x},\, \mathbf{y}\rangle = 0$
- $\|\mathbf{x}\| = 1$
- $\|\mathbf{y}\| = 1$

In other words, $\{\mathbf{x},\, \mathbf{y}\}$ is an orrthonormal set (Definition ONS) and so by Theorem CUMOS, B is a unitary matrix.

4. For the matrix A below, in each part of this question determine a set S so that the column space of A is the span of S, $\mathcal{C}(A) = \langle S\rangle$, and S has the required properties obtained by the indicated methods. (40 points)

$$A = \begin{bmatrix} -1 & 0 & 1 & -2 & -1 \\ 0 & -1 & 1 & -1 & -3 \\ -4 & 3 & 1 & -5 & 5 \\ 3 & -2 & -1 & 4 & -3 \end{bmatrix}$$

(a) S has 5 elements and illustrates the definition of the column space, Definition CSM.

Solution: The column space is the span of the columns of the matrix (Definition CSM). So

$$S = \left\{ \begin{bmatrix} -1 \\ 0 \\ -4 \\ 3 \end{bmatrix},\, \begin{bmatrix} 0 \\ -1 \\ 3 \\ -2 \end{bmatrix},\, \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix},\, \begin{bmatrix} -2 \\ -1 \\ -5 \\ 4 \end{bmatrix},\, \begin{bmatrix} -1 \\ -3 \\ 5 \\ -3 \end{bmatrix} \right\}$$

(b) S is linearly independent, and illustrates Theorem BCS, Basis of Column Space.

Solution: Theorem BCS is a rehash of Theorem BS: row-reduce the matrix, identify indices of pivot columns and use the columns of the original matrix with the same indices.

$$\begin{bmatrix} -1 & 0 & 1 & -2 & -1 \\ 0 & -1 & 1 & -1 & -3 \\ -4 & 3 & 1 & -5 & 5 \\ 3 & -2 & -1 & 4 & -3 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & -1 & 2 & 1 \\ 0 & \boxed{1} & -1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

So $D = \{1,\, 2\}$ and

$$S = \left\{ \begin{bmatrix} -1 \\ 0 \\ -4 \\ 3 \end{bmatrix},\, \begin{bmatrix} 0 \\ -1 \\ 3 \\ -2 \end{bmatrix} \right\}$$

(c) S is linearly independent, and illustrates Theorem BRS, Basis of Row Space.

Solution: The column space of A is the row space of A^t (Theorem CSRST). So transpose A, row reduce, and by Theorem BRS select the non-zero rows as columns vectors in S.

$$A^t = \begin{bmatrix} -1 & 0 & -4 & 3 \\ 0 & -1 & 3 & -2 \\ 1 & 1 & 1 & -1 \\ -2 & -1 & -5 & 4 \\ -1 & -3 & 5 & -3 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 4 & -3 \\ 0 & \boxed{1} & -3 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 4 \\ -3 \end{bmatrix},\, \begin{bmatrix} 0 \\ 1 \\ -3 \\ 2 \end{bmatrix} \right\}$$

(d) S is linearly independent, and illustrates Theorem FS, Four Subsets.

Solution: We form the extended echelon form of the matrix,

$$M = \begin{bmatrix} -1 & 0 & 1 & -2 & -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 & -3 & 0 & 1 & 0 & 0 \\ -4 & 3 & 1 & -5 & 5 & 0 & 0 & 1 & 0 \\ 3 & -2 & -1 & 4 & -3 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & -1 & 2 & 1 & 0 & 0 & 2 & 3 \\ 0 & \boxed{1} & -1 & 1 & 3 & 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & 3 & 4 \end{bmatrix}$$

The last two rows, in the last four columns, form the matrix L, which is already in reduced row-echelon form

$$L = \begin{bmatrix} \boxed{1} & 0 & 2 & 3 \\ 0 & \boxed{1} & 3 & 4 \end{bmatrix}$$

and by Theorem FS, the null space equal of L is equal to the column space of A, so we can apply Theorem BNS,

$$\mathcal{C}(A) = \mathcal{N}(L) = \left\langle \left\{ \begin{bmatrix} -2 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -4 \\ 0 \\ 1 \end{bmatrix} \right\} \right\rangle$$

5. Using the matrix A from the previous problem, find a column vector $\mathbf{b}$ so that $\mathcal{LS}(A, \mathbf{b})$ is consistent and the first two entries of $\mathbf{b}$ are 2 and 5 (more precisely, $[\mathbf{b}]_1 = 2$ and $[\mathbf{b}]_2 = 5$). (10 points)

   Solution: We desire $\mathbf{b}$ to be in the column space of A to form a consistent system (Theorem CSCS). So any linear combination of the vectors in the various versions of S above will lead to a consistent system. To meet the condition on the first two entries, the S in part (c) will work best, but strictly by accident, the S in (b) is not too far off.

   The unique answer to this question is

$$\mathbf{b} = 2 \begin{bmatrix} 1 \\ 0 \\ 4 \\ -3 \end{bmatrix} + 5 \begin{bmatrix} 0 \\ 1 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ -7 \\ 4 \end{bmatrix}$$

   Can you write $\mathbf{b}$ as a linear combionation of the elements of S from part (d)? How many ways?

6. Suppose that A and B are $m \times n$ matrices. Write a careful proof that $\overline{A+B} = \overline{A} + \overline{B}$. (15 points)

   Solution: This is Theorem CRMA. See the proof there.

7. Suppose that A is an $m \times n$ matrix and I_m is the size m identity matrix. Write a careful proof that $I_m A = A$. (15 points)

   Solution: This is Theorem MMIM. See a similar proof there.