Math 290 Name: Key Dr. Beezer
Quiz VS Fall 2007

Show *all* of your work and *explain* your answers fully. There is a total of 100 possible points. If you use a calculator or Mathematica on a problem be sure to write down both the input and output.

1. In the vector space of 2×2 matrices, M_{22}, determine if the set

$$P = \left\{ \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}, \begin{bmatrix} -3 & 0 \\ 2 & -2 \end{bmatrix}, \begin{bmatrix} -1 & 4 \\ 2 & 6 \end{bmatrix} \right\}$$

is linearly independent. (15 points)

Solution: Begin with a relation of linear dependence (Definition RLD) on P with three unknown scalars, and determine if they can must be trivial, or if there are non-trivial possibilities,

$$\begin{aligned} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} &= a \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix} + b \begin{bmatrix} -3 & 0 \\ 2 & -2 \end{bmatrix} + c \begin{bmatrix} -1 & 4 \\ 2 & 6 \end{bmatrix} \\ &= \begin{bmatrix} 2a - 3b - c & a + 4c \\ -a + 2b + 2c & 3a - 2c + 6c \end{bmatrix} \end{aligned}$$

The definition of matrix equality (Definition ME) provides a homogeneous system of 4 linear equation in 3 variables,

$$\begin{aligned} 2a - 3b - c &= 0 \\ a + 4c &= 0 \\ -a + 2b + 2c &= 0 \\ 3a - 2c + 6c &= 0 \end{aligned}$$

Solving this system leads to non-trivial solutions (such as $a = -4$, $b = -3$, $c = 1$) which provide non-trivial relations of linear dependence on P, so by Definition LI, the set P is linearly dependent.

2. In the vector space of polynomials of degree at most 2, P_2, does the set

$$Q = \left\{1 + 3x - x^2,\ 2 + x + 3x^2,\ -1 + 4x^2\right\}$$

span P_2? (15 points)

Solution: Can we write an arbitrary polynomial in P_2, say $a + bx + cx^2$ as a linear combination of the elements of Q? Equivalently, can we find scalars, a_1, a_2, a_3 so that

$$\begin{aligned} a + bx + cx^2 &= a_1(1 + 3x - x^2) + a_2(2 + x + 3x^2) + a_3(-1 + 4x^2) \\ &= (a_1 + 2a_2 - a_3) + (3a_1 + a_2)x + (-a_1 + 3a_2 + 4a_3)x^2 \end{aligned}$$

Equating coefficients, according to the definition of equality in P_2, we arrive at a system of three equations in the three variables a_1, a_2, a_3, $\mathcal{LS}(A,\ \mathbf{b})$, where

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & 0 \\ -1 & 3 & 4 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Since A row-reduces to the identity matrix I_3, by Theorem NMRRI, A is nonsingular, and Theorem NMUS tells us this system always has a (unique) solution. So we can alwys find the scalars a_1, a_2, a_3 and so $\langle Q \rangle = P_2$ and by Definition TSVS, Q spans P_2. (Another approach is to establish the linear independence of Q and then the known dimension of P_2 allows us to apply Theorem G to establish the spanning property.)

3. In the vector space of polynomials with degree at most 2, P_2, the set

$$X = \left\{ a + bx + cx^2 \,\middle|\, a - 2b + 5c = 0 \right\}$$

is a subspace of P_2. (You may assume these statements so far and **do not** need to prove them.) Find the dimension of X (with proof). (15 points)

Solution: We must find a basis of X in order to determine the dimension. We first will try to find a spanning set and hope for its linear independence,

$$\begin{aligned}
X &= \left\{ a + bx + cx^2 \,\middle|\, a - 2b + 5c = 0 \right\} \\
&= \left\{ a + bx + cx^2 \,\middle|\, a = 2b - 5c \right\} \\
&= \left\{ (2b - 5c) + bx + cx^2 \,\middle|\, b,\, c \in \mathbb{C} \right\} \\
&= \left\{ (2b + bx) + \left(-5c + cx^2\right) \,\middle|\, b,\, c \in \mathbb{C} \right\} \\
&= \left\{ b\,(2 + x) + c\left(-5 + x^2\right) \,\middle|\, b,\, c \in \mathbb{C} \right\} \\
&= \left\langle \left\{ 2 + x,\, -5 + x^2 \right\} \right\rangle
\end{aligned}$$

So $P = \left\{ 2 + x,\, -5 + x^2 \right\}$ is a spanning set for X (Definition TSVS). Is P linearly independent? Begin with a relation of linear dependence (Definition RLD),

$$\begin{aligned}
0 + 0x + 0x^2 &= a_1(2 + x) + a_2(-5 + x^2) \\
&= (2a_1 - 5a_2) + a_1 x + a_2 x^2
\end{aligned}$$

Equality of polynomials (Example VSP) implies that $a_1 = a_2 = 0$, so by Definition LI, P is linearly independent. Definition B says that P is a basis of X, and finally, Definition D, tells us that X has dimension 2.

4. In the vector space of 2×2 matrices, M_{22}, the set

$$W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \,\middle|\, a - 3b + c - 4d = 0 \right\}$$

is a subspace with dimension 3. (You may assume these statements so far and **do not** need to prove them.) For each set of 2×2 matrices below, determine if they have the indicated properties. (25 points)

(a) Does R span W? $\quad R = \left\{ \begin{bmatrix} 6 & 1 \\ 1 & 1 \end{bmatrix},\, \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix} \right\}$

Solution: Since the size of R is less than the dimension of W, Theorem G tells us that R does not span W.

(b) Is T a basis of W? $\quad T = \left\{ \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix},\, \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix},\, \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

Solution: Any basis of W must have size 3 (Theorem BIS), so T cannot be ruled out as a basis based on size alone. We first check linear independence of T with a relation of linear dependence (Definition RLD),

$$\begin{aligned}
\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} &= a_1 \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix} + a_2 \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} + a_3 \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 3a_1 - a_2 + 4a_3 & a_1 \\ a_2 & a_3 \end{bmatrix}
\end{aligned}$$

Applying the definition of matrix equality (Definition ME), we conclude that $a_1 = a_2 = a_3 = 0$. So by Definition LI, T is linearly independent. Then, along with the size of T equaling the dimension of W, Theorem G tells us that T spans W. Finally, Definition B is fulfilled and T is a basis of W.

(c) Does S span W? $\quad S=\left\{\begin{bmatrix}0 & 2\\ 2 & -1\end{bmatrix},\ \begin{bmatrix}8 & 15\\ 0 & 0\end{bmatrix},\ \begin{bmatrix}3 & 1\\ 4 & 1\end{bmatrix}\right\}$

Solution: The matrix $\begin{bmatrix}8 & 15\\ 0 & 0\end{bmatrix}$ is not an element of W, so the span of S, $\langle S\rangle$ will contain elements of M_{22} that are not in W. Thus, S does not span W.

5. Suppose that $S=\{\mathbf{v}_1,\,\mathbf{v}_2\}$ is a linearly independent set in the vector space V. Prove that $T=\{4\mathbf{v}_1+\mathbf{v}_2,\,5\mathbf{v}_1+\mathbf{v}_2\}$ is a linearly independent set in V. (15 points)

Solution: Begin with a relation of linear dependence on T with the aim of determining that the two unknown scalars *must* both be zero.

$$\begin{aligned}
\mathbf{0} &= a_1\left(4\mathbf{v}_1+\mathbf{v}_2\right)+a_2\left(5\mathbf{v}_1+\mathbf{v}_2\right) && \text{Definition RLD}\\
&= 4a_1\mathbf{v}_1+a_1\mathbf{v}_2+5a_2\mathbf{v}_1+a_2\mathbf{v}_2 && \text{Property DVA}\\
&= 4a_1\mathbf{v}_1+5a_2\mathbf{v}_1+a_1\mathbf{v}_2+a_2\mathbf{v}_2 && \text{Property AC}\\
&= \left(4a_1+5a_2\right)\mathbf{v}_1+\left(a_1+a_2\right)\mathbf{v}_2 && \text{Property DSA}
\end{aligned}$$

We now have a relation of linear dependence on S (Definition RLD), which is a set of vectors assumed to be linearly independent, so by Definition LI we conclude that the two scalars in the linear combination are both zero,

$$\begin{aligned}
4a_1+5a_2&=0\\
a_1+a_2&=0
\end{aligned}$$

This is a homogeneous system with a nonsingular coefficient matrix (check this!), so by Theorem NMUS, the solution is unique and must be the trivial solution $a_1=0$, $a_2=0$. By Definition LI we conclude that T is linearly independent.

6. Suppose that $\mathbf{v}$ is an element of the vector space V, and $\alpha\in\mathbb{C}$. then Theorem AISM tells us that $(-1)\mathbf{v}=-\mathbf{v}$. Prove the more general statement that $(-\alpha)\mathbf{v}=-(\alpha\mathbf{v})$. (15 points)

Solution:

$$\begin{aligned}
(-\alpha)\mathbf{v} &= \left((-1)\alpha\right)\mathbf{v} && \text{Arithmetic in }\mathbb{C}\\
&= (-1)\left(\alpha\mathbf{v}\right) && \text{Property SMA}\\
&= -\left(\alpha\mathbf{v}\right) && \text{Theorem AISM}
\end{aligned}$$