Show *all* of your work and *explain* your answers fully. There is a total of 130 possible points. If you use a calculator or Mathematica on a problem be sure to write down both the input and output. Give complete explanations, even for negative answers (such as "no" or "impossible.").

1. Analyze the linear transformation $S\colon \mathbb{C}^3 \mapsto \mathbb{C}^4$, $S\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right) = \begin{bmatrix}-x_2+3x_3\\x_1+3x_2-7x_3\\x_1+x_2-x_3\\x_1+2x_2-4x_3\end{bmatrix}$. (45 points)

   (a) Determine a basis for the kernel of S, $\mathcal{K}(S)$.

   Solution: The condition that $S(\mathbf{x}) = \mathbf{0}$ leads to a homogeneous system of 4 equations in 3 variables. The coefficient matrix of this system row-reduces as,

$$\begin{bmatrix}0 & -1 & 3\\1 & 3 & -7\\1 & 1 & -1\\1 & 2 & -4\end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix}\boxed{1} & 0 & 2\\0 & \boxed{1} & -3\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$$

   Applying Theorem BNS, we have a basis for the solution set of the system, which is the kernel of the linear transformation,

$$\mathcal{K}(S) = \left\langle\left\{\begin{bmatrix}-2\\3\\1\end{bmatrix}\right\}\right\rangle$$

   (b) Determine a basis for the range of S, $\mathcal{R}(S)$.

   Solution: By Theorem SSRLT, we can write the range as the span of selected outputs of S, when evaluated at the elements of a basis for the domain (in this case the basis of standard unit vectors, Theorem SUVB).

$$\begin{aligned}\mathcal{R}(S) &= \langle\{S(\mathbf{e}_1), S(\mathbf{e}_2), S(\mathbf{e}_3)\}\rangle\\ &= \left\langle\left\{\begin{bmatrix}0\\1\\1\\1\end{bmatrix}, \begin{bmatrix}-1\\3\\1\\2\end{bmatrix}, \begin{bmatrix}3\\-7\\-1\\-4\end{bmatrix}\right\}\right\rangle\end{aligned}$$

   We can convert this spanning set into a basis with applications of Theorem CSRST and Theorem BRS which only require row-reducing a matrix with the column vectors of the spanning set as rows of the matrix,

$$\begin{bmatrix}0 & 1 & 1 & 1\\-1 & 3 & 1 & 2\\3 & -7 & -1 & -4\end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix}1 & 0 & 2 & 1\\0 & 1 & 1 & 1\\0 & 0 & 0 & 0\end{bmatrix}$$

   Thus

$$\mathcal{R}(S) = \left\langle\left\{\begin{bmatrix}1\\0\\2\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\\1\end{bmatrix}\right\}\right\rangle$$

(c) Is S injective?

Solution: No, by Theorem KILT.

(d) Is S surjective?

Solution: No, by Theorem RSLT.

(e) Is S invertible?

Solution: No, by Theorem ILTIS.

(f) If possible, find two different vectors, $\mathbf{x}$, $\mathbf{y} \in \mathbb{C}^3$ such that $S(\mathbf{x}) = S(\mathbf{y})$.

Solution: Choose any vector for $\mathbf{x}$ (even the zero vector), say $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Then find any non-trivial element of the kernel (we'll use the basis vector above), say $\mathbf{z} = \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$, and then

$$\mathbf{y} = \mathbf{x} + \mathbf{z} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \\ 4 \end{bmatrix}$$

will meet the requirements of the question.

(You can check here that S carries both $\mathbf{x}$ and $\mathbf{y}$ to the output $\begin{bmatrix} 7 \\ -14 \\ 0 \\ -7 \end{bmatrix}$).

(g) If possible, find a vector, $\mathbf{w} \in \mathbb{C}^3$ such that $S(\mathbf{w}) = \begin{bmatrix} 2 \\ -1 \\ 3 \\ 4 \end{bmatrix}$

Solution: This is only possible if the vector $\mathbf{r} = \begin{bmatrix} 2 \\ -1 \\ 3 \\ 4 \end{bmatrix}$ is an element of the range of S. Using the basis for $\mathcal{R}(S)$ from part (b) we see that every vector in $\mathcal{R}(S)$ whose first two entries are 2 and -1 must be

$$2 \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix} + (-1) \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \\ 1 \end{bmatrix} \neq \mathbf{r}$$

So this is impossible. In other words, $\mathbf{r}$ lies outside of the range of S, or equivalently, $\mathbf{r}$ has an empty pre-image. (Note: this question could also be answered by analyzing an inconsistent system of equations via an augmented matrix.)

(h) Based only on your analysis above, describe the possibilities for the pre-image of $\begin{bmatrix} 1 \\ 1 \\ 3 \\ 2 \end{bmatrix}$, $S^{-1}\left(\begin{bmatrix} 1 \\ 1 \\ 3 \\ 2 \end{bmatrix}\right)$.

Solution: The pre-image could be empty, as in part (g). Or the pre-image could be non-empty, and Theorem KPI, together with the non-trivial kernel in part (a) will yield an infinite pre-image.

(i) Now compute the pre-image of $\begin{bmatrix}1\\1\\3\\2\end{bmatrix}$, $S^{-1}\left(\begin{bmatrix}1\\1\\3\\2\end{bmatrix}\right)$.

Solution: We will find one vector $\mathbf{w} \in \mathbb{C}^3$ such that $S(\mathbf{w}) = \begin{bmatrix}1\\1\\3\\2\end{bmatrix}$. This condition leads to a system of equations with an augmented matrix that row-reduces as

$$\begin{bmatrix}0 & -1 & 3 & 1\\ 1 & 3 & -7 & 1\\ 1 & 1 & -1 & 3\\ 1 & 2 & -4 & 2\end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix}\boxed{1} & 0 & 2 & 4\\ 0 & \boxed{1} & -3 & -1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$$

To apply Theorem KPI we only need a single element of the pre-image, so we can solve the system above with the single free variable set equal to zero to obtain,

$$\begin{aligned}S^{-1}\left(\begin{bmatrix}1\\1\\3\\2\end{bmatrix}\right) &= \mathbf{w} + \mathcal{K}(T)\\ &= \begin{bmatrix}4\\-1\\0\end{bmatrix} + \left\langle\left\{\begin{bmatrix}-2\\3\\1\end{bmatrix}\right\}\right\rangle\\ &= \left\{\begin{bmatrix}4\\-1\\0\end{bmatrix} + \alpha\begin{bmatrix}-2\\3\\1\end{bmatrix} \;\middle|\; \alpha \in \mathbb{C}\right\}\end{aligned}$$

The same result can be found with the vector form of the solutions, (Theorem VFSLS) to the system above.

2. Analyze the linear transformation $T\colon \mathbb{C}^3 \mapsto \mathbb{C}^3$, $\quad T\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right) = \begin{bmatrix}2x_1 - x_2\\ 2x_1 - x_2 + x_3\\ x_1 - x_2 - x_3\end{bmatrix}$. (45 points)

(a) Determine the kernel of T, $\mathcal{K}(T)$.

Solution: The condition that $T(\mathbf{x}) = \mathbf{0}$ leads to a homogeneous system of 3 equations in 3 variables. The coefficient matrix of this system row-reduces as,

$$\begin{bmatrix}2 & -1 & 0\\ 2 & -1 & 1\\ 1 & -1 & -1\end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix}\boxed{1} & 0 & 0\\ 0 & \boxed{1} & 0\\ 0 & 0 & \boxed{1}\end{bmatrix}$$

With only the trivial solution, the kernel is trivial, $\mathcal{K}(T) = \{\mathbf{0}\}$.

(b) Determine the range of T, $\mathcal{R}(T)$.

Solution: An application of Theorem RPNDD yields,

$$\dim \mathcal{R}(T) = \dim \mathbb{C}^3 - \dim \mathcal{N}(T) = 3 - 0 = 3$$

So $\mathcal{R}(T)$ is a dimension 3 subspace of $\mathbb{C}^3$, and by Theorem EDYES, we have $\mathcal{R}(T) = \mathbb{C}^3$.

(c) Is T injective?

Solution: Yes, by Theorem KILT.

(d) Is T surjective?

Solution: Yes, by Theorem RSLT.

(e) Is T invertible?

Solution: Yes, by Theorem ILTIS.

(f) If possible, find two different vectors, $\mathbf{x}, \mathbf{y} \in \mathbb{C}^3$ such that $T(\mathbf{x}) = T(\mathbf{y})$.

Solution: This would violate the definition of an injective linear transformation, Definition ILT.

(g) If possible, find a vector, $\mathbf{w} \in \mathbb{C}^3$ such that $T(\mathbf{w}) = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$.

Solution: Since T is a surjective linear transformation, this is possible for any choice of the codomain element. The condition that $T(\mathbf{w}) = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$ leads to a system of 3 equations in 3 variables with an augmented matrix that row-reduces as,

$$\begin{bmatrix} 2 & -1 & 0 & 2 \\ 2 & -1 & 1 & -1 \\ 1 & -1 & -1 & 4 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 0 & 1 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & -3 \end{bmatrix}$$

So the unique solution for $\mathbf{w}$ is

$$\mathbf{w} = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}$$

(h) Based only on your analysis above, describe the possibilities for the pre-image of $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $T^{-1}\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right)$.

Solution: An invertible linear transformation will have a pre-image that is a set of size 1 for any chosen element of the codomain.

(i) Now compute the pre-image of $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $T^{-1}\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right)$.

Solution: In a fashion very similar to part (g) the requirement for membership in the pre-image produces a system of equations with a unique solution,

$$\begin{bmatrix} 2 & -1 & 0 & 1 \\ 2 & -1 & 1 & 0 \\ 1 & -1 & -1 & 1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 0 & 1 \\ 0 & \boxed{1} & 0 & 1 \\ 0 & 0 & \boxed{1} & -1 \end{bmatrix}$$

and

$$T^{-1}\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right) = \left\{\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}\right\}$$

3. The linear transformation $Q\colon \mathbb{C}^2 \mapsto P_1$ defined below is invertible (you can assume this). Find a formula for the inverse linear transformation $Q^{-1}\colon P_1 \mapsto \mathbb{C}^2$. (20 points)
(P₁ is the vector space of polynomials with degree at most 1.)

$$Q\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = (3a+b) + (2a+b)\,x$$

Solution: We begin with a basis for the domain of Q^{-1}, which is the codomain of Q, namely P_1. A good choice for such a basis is $B = \{1,\, x\}$. If we understand how Q^{-1} behaves on this basis, then we understand it entirely (Theorem LTDB). We can compute the values of $Q^{-1}(1)$ and $Q^{-1}(x)$ as the unique elements of the pre-images, $Q^{-1}(1)$ and $Q^{-1}(x)$.

Membership for $\begin{bmatrix} a \\ b \end{bmatrix}$ in $Q^{-1}(1)$ leads to a system of two equations in two variables with an augmented matrix that row-reduces as

$$\begin{bmatrix} 3 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 1 \\ 0 & \boxed{1} & -2 \end{bmatrix}$$

So $Q^{-1}(1) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

Similarly, membership for $\begin{bmatrix} a \\ b \end{bmatrix}$ in $Q^{-1}(x)$ leads to a system of two equations in two variables with an augmented matrix that row-reduces as

$$\begin{bmatrix} 3 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & -1 \\ 0 & \boxed{1} & 3 \end{bmatrix}$$

So $Q^{-1}(x) = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$.

With an understanding of Q^{-1} on a basis we can concoct a general expression,

$$\begin{aligned} Q^{-1}(c+dx) &= cQ^{-1}(1) + dQ^{-1}(x) && \text{Theorem LTLC} \\ &= c\begin{bmatrix} 1 \\ -2 \end{bmatrix} + d\begin{bmatrix} -1 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} c-d \\ -2c+3d \end{bmatrix} \end{aligned}$$

4. Suppose that $T\colon U \mapsto V$ is a linear transformation and $\mathbf{x}$ and $\mathbf{y}$ are two different elements of the pre-image $T^{-1}(\mathbf{w})$. Prove that $\mathbf{x} + \mathcal{K}(T) = \mathbf{y} + \mathcal{K}(T)$. (Theorem KPI says we can write a non-empty preimage as an arbitrary element of the preimage plus elements of the kernel. Do not use this theorem to answer this question, but instead establish the conclusion directly as a way of explaining why Theorem KPI allows total freedom in the choice of the pre-image element.) (20 points)

Solution: The conclusion is an equality of sets, so we proceed according to Definition SE. Choose an arbitrary element of $\mathbf{x}+\mathcal{K}(T)$, which will be a vector of the form $\mathbf{x}+\mathbf{z}$ where $\mathbf{z} \in \mathcal{K}(T)$. Define $\mathbf{v} = \mathbf{x}-\mathbf{y}+\mathbf{z}$ and note that

$$\begin{aligned} T(\mathbf{v}) &= T(\mathbf{x}-\mathbf{y}+\mathbf{z}) \\ &= T(\mathbf{x}) - T(\mathbf{y}) + T(\mathbf{z}) && \text{Theorem LTLC} \\ &= \mathbf{w} - \mathbf{w} + T(\mathbf{z}) && \text{Definition PI} \\ &= \mathbf{0} + T(\mathbf{z}) && \text{Property AI} \\ &= T(\mathbf{z}) && \text{Property Z} \\ &= \mathbf{0} && \text{Definition KLT} \end{aligned}$$

So $\mathbf{v} \in \mathcal{K}(T)$ by Definition KLT.

Also,

$$\begin{aligned} \mathbf{x} + \mathbf{z} &= \mathbf{y} + \mathbf{x} - \mathbf{y} + \mathbf{z} \\ &= \mathbf{y} + \mathbf{v} \end{aligned}$$

and we see that $\mathbf{x} + \mathbf{z}$ qualifies for membership in $\mathbf{y} + \mathcal{K}(T)$. So $\mathbf{x} + \mathcal{K}(T) \subseteq \mathbf{y} + \mathcal{K}(T)$.

By an entirely similar argument, we can get the opposite inclusion to complete the requirements of Definition SE for set equality.