Math 290 — Name: Key — Dr. Beezer

Quiz V — Spring 2007

Show *all* of your work and *explain* your answers fully. There is a total of 100 possible points. If you use a calculator or Mathematica on a problem be sure to write down both the input and output.

1. Find all solutions to the system of equations below, writing the solutions as a set employing the vector form of a solution. (10 points)

$$\begin{aligned} -2x_1 - 6x_2 + x_3 + 4x_4 + 9x_5 &= 7 \\ 3x_1 + 9x_2 + 5x_3 + 7x_4 - 7x_5 &= 9 \end{aligned}$$

Solution: Theorem VFSLS references the row-reduced version of the augmented matrix of the system. So we form the augmented matrix and find a row-equivalent matrix in reduced row-echelon form,

$$\begin{bmatrix} -2 & -6 & 1 & 4 & 9 & 7 \\ 3 & 9 & 5 & 7 & -7 & 9 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 3 & 0 & -1 & -4 & -2 \\ 0 & 0 & \boxed{1} & 2 & 1 & 3 \end{bmatrix}$$

The non-pivot columns have indices $F = \{2, 4, 5\}$. Based on this alone, we know a typical solution vector looks like

$$\mathbf{x} = \begin{bmatrix} \\ 0 \\ \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} \\ 1 \\ \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} \\ 0 \\ \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} \\ 0 \\ \\ 0 \\ 1 \end{bmatrix}$$

Applying the theorem, or unraveling the equations represented by each row of the augmented matrix, we can use the vector form of a solution vector in a set construction to achieve the solution set,

$$\left\{ \begin{bmatrix} -2 \\ 0 \\ 3 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 4 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \,\middle|\, x_2,\, x_4,\, x_5 \in \mathbb{C} \right\}$$

2. For the matrix A below, find a linearly independent set P such that $\langle P \rangle = \mathcal{N}(A)$, i.e. the span of P equals the null space of A. (15 points)

$$A = \begin{bmatrix} -2 & -4 & -4 & 10 & -4 \\ -4 & -1 & 5 & 1 & 24 \\ 2 & -1 & -4 & 1 & -15 \end{bmatrix}$$

Solution: Theorem BNS provides the necesary vectors. First, we need to row-reduce the matrix to find a row-equivalent matrix in reduced row-echelon form,

$$\begin{bmatrix} -2 & -4 & -4 & 10 & -4 \\ -4 & -1 & 5 & 1 & 24 \\ 2 & -1 & -4 & 1 & -15 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 0 & -3 & -2 \\ 0 & \boxed{1} & 0 & 1 & -1 \\ 0 & 0 & \boxed{1} & -2 & 3 \end{bmatrix}$$

The non-pivot columns have indices $F = \{4, 5\}$, and with an application of Theorem BNS, or consideration of solutions to the homogeneous system, $\mathcal{LS}(A, \mathbf{0})$, we find the linear independent set

$$P = \left\{ \begin{bmatrix} 3 \\ -1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

whose span will be the null space of A.

3. Given the set S below, find a linearly independent set T such that $\langle T \rangle = \langle S \rangle$. (15 points)

$$S = \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 5 \\ -4 \\ 1 \end{bmatrix}, \begin{bmatrix} -4 \\ 11 \\ -10 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ 6 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ 7 \\ -13 \\ 7 \end{bmatrix} \right\}$$

Solution: Theorem BS provides the tools for locating such a set. Let C be the matrix whose columns are the vectors in S, and then row-reduce,

$$C = \begin{bmatrix} 1 & -2 & -4 & 1 & -1 \\ -2 & 5 & 11 & -4 & 7 \\ 1 & -4 & -10 & 6 & -13 \\ 0 & 1 & 3 & -3 & 7 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 2 & 0 & 3 \\ 0 & \boxed{1} & 3 & 0 & 1 \\ 0 & 0 & 0 & \boxed{1} & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The pivot columns are $D = \{1, 2, 4\}$ so we can form the set T as simply the first three vectors of S, and know from Theorem BS that T is linearly independent and $\langle T \rangle = \langle S \rangle$,

$$T = \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 5 \\ -4 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ 6 \\ -3 \end{bmatrix} \right\}$$

4. For each set below, determine if the set is linearly independent or linearly dependent. (15 points)

(a) $$T = \left\{ \begin{bmatrix} 5 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \\ 7 \\ 2 \end{bmatrix}, \begin{bmatrix} 6 \\ -2 \\ 2 \\ -2 \end{bmatrix} \right\}$$

Solution: With 4 vectors from $\mathbb{C}^4$ we can make these vectors the columns of a matrix, B, and row-reduce the result,

$$B = \begin{bmatrix} 5 & -2 & -1 & 6 \\ 1 & 3 & 4 & -2 \\ 1 & 5 & 7 & 2 \\ 1 & 1 & 2 & -2 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & \boxed{1} \end{bmatrix}$$

Since the row-reduced version of B is the 4×4 identity matrix, I_4 (Definition IM), we know by Theorem NMRI that the columns of B are linearly independent. Thus T is a linearly independent set.

(b) $$R = \left\{ \begin{bmatrix} -2 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 8 \end{bmatrix}, \begin{bmatrix} 7 \\ 5 \\ 7 \end{bmatrix}, \begin{bmatrix} -2 \\ 6 \\ 1 \end{bmatrix} \right\}$$

Solution: This is a set of $n = 5$ vectors from $\mathbb{C}^3$ and $n = 5 > 3 = m$, so by Theorem MVSLD the set R is linearly dependent.

5. Suppose that $\alpha \in \mathbb{C}$ is a scalar and $\mathbf{x} \in \mathbb{C}^m$ is a vector. Prove that $\overline{\alpha\mathbf{x}} = \overline{\alpha}\,\overline{\mathbf{x}}$ with a careful proof that uses the definition of vector equality (Definition CVE). (15 points)

   Solution: For $1 \leq i \leq m$,

$$\begin{aligned}
[\overline{\alpha\mathbf{x}}]_i &= \overline{[\alpha\mathbf{x}]_i} && \text{Definition CCCV} \\
&= \overline{\alpha\,[\mathbf{x}]_i} && \text{Definition CVSM} \\
&= \overline{\alpha}\,\overline{[\mathbf{x}]_i} && \text{Definition CCRM} \\
&= \overline{\alpha}\ [\overline{\mathbf{x}}]_i && \text{Definition CCCV}
\end{aligned}$$

   Then by Definition CVE we have $\overline{\alpha\mathbf{x}} = \overline{\alpha}\,\overline{\mathbf{x}}$.

6. Suppose that $\mathbf{u},\, \mathbf{v} \in \mathbb{C}^m$ are two vectors. Use the definition of linear independence to prove that $S = \{\mathbf{u},\, \mathbf{v}\}$ is a linearly dependent set if and only if one of the two vectors is a scalar multiple of the other. In other words, do not simply apply Theorem DLDS, the theorem that says linear dependence is equivalent to one vector being a linear combination of the others. (15 points)

   Solution: ($\Rightarrow$) If S is linearly dependent, then there are scalars α and β, not both zero, such that $\alpha\mathbf{u} + \beta\mathbf{v} = \mathbf{0}$. Suppose that $\alpha \neq 0$, the proof proceeds similarly if $\beta \neq 0$. Now,

$$\begin{aligned}
\mathbf{u} &= 1\mathbf{u} && \text{Property OC} \\
&= \left(\frac{1}{\alpha}\alpha\right)\mathbf{u} && \text{Property MICN} \\
&= \frac{1}{\alpha}(\alpha\mathbf{u}) && \text{Property SMAC} \\
&= \frac{1}{\alpha}(\alpha\mathbf{u} + \mathbf{0}) && \text{Property ZC} \\
&= \frac{1}{\alpha}(\alpha\mathbf{u} + \beta\mathbf{v} - \beta\mathbf{v}) && \text{Property AIC} \\
&= \frac{1}{\alpha}(\mathbf{0} - \beta\mathbf{v}) && \text{Definition LICV} \\
&= \frac{1}{\alpha}(-\beta\mathbf{v}) && \text{Property ZC} \\
&= \frac{-\beta}{\alpha}\mathbf{v} && \text{Property SMAC}
\end{aligned}$$

   which shows that $\mathbf{u}$ is a scalar multiple of $\mathbf{v}$.

   ($\Leftarrow$) Suppose now that $\mathbf{u}$ is a scalar multiple of $\mathbf{v}$. More precisely, suppose there is a scalar γ such that $\mathbf{u} = \gamma\mathbf{v}$. Then

$$\begin{aligned}
(-1)\mathbf{u} + \gamma\mathbf{v} &= (-1)\mathbf{u} + \mathbf{u} && \\
&= (-1)\mathbf{u} + (1)\mathbf{u} && \text{Property OC} \\
&= ((-1) + 1)\,\mathbf{u} && \text{Property DSAC} \\
&= 0\mathbf{u} && \text{Property AICN} \\
&= \mathbf{0} && \text{Definition CVSM}
\end{aligned}$$

   This is a relation of linear of linear dependence on S (Definition RLDCV), which is nontrivial since one of the scalars is -1. Therefore S is linearly dependent by Definition LICV.