Math 290 Name: Key Dr. Beezer
Quiz V Spring 2007

Show *all* of your work and *explain* your answers fully. There is a total of 100 possible points. If you use a calculator or Mathematica on a problem be sure to write down both the input and output.

1. Find all solutions to the system of equations below, using the inverse of the coefficient matrix. Demonstrate how to find the inverse by a procedure using row operations to obtain the extended echelon form of the matrix. (15 points)

$$\begin{aligned} x_1 + 3x_2 - x_3 &= -4 \\ -x_1 - 2x_2 + 3x_3 &= 8 \\ 2x_1 + 5x_2 - 3x_3 &= -9 \end{aligned}$$

Solution: By Theorem SLEMM we can view this linear system as a vector equation of the form $A\mathbf{x} = \mathbf{b}$. Attach the 3×3 identity matrix to the coefficient matrix A and row-reduce,

$$\begin{bmatrix} 1 & 3 & -1 & 1 & 0 & 0 \\ -1 & -2 & 3 & 0 & 1 & 0 \\ 2 & 5 & -3 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 0 & -9 & 4 & 7 \\ 0 & \boxed{1} & 0 & 3 & -1 & -2 \\ 0 & 0 & \boxed{1} & -1 & 1 & 1 \end{bmatrix}$$

So by Theorem CINM we have the inverse of the coefficent matrix as

$$A^{-1} = \begin{bmatrix} -9 & 4 & 7 \\ 3 & -1 & -2 \\ -1 & 1 & 1 \end{bmatrix}$$

We recognize A as a nonsingular matrix (Theorem NI), so we can apply Theorem SNCM to obtain the unique solution to this system as the vector

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} -9 & 4 & 7 \\ 3 & -1 & -2 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -4 \\ 8 \\ -9 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 3 \end{bmatrix}$$

2. For the matrix A below, find a linearly independent set of vectors S whose span equals the column space of A, that is $\langle S\rangle = \mathcal{C}(A)$. The two parts of this problem ask for different versions of the set S. (20 points)

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -4 & 1 & -3 & -7 \\ 1 & -7 & -6 & -5 \end{bmatrix}$$

(a) Construct S so that the vectors are also columns of A.

Solution: In anticipation of using Theorem BCS, we row-reduce A,

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ -4 & 1 & -3 & -7 \\ 1 & -7 & -6 & -5 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 1 & 2 \\ 0 & \boxed{1} & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So the set of pivot columns is $D = \{1,\ 2\}$, so Theorem BCS says we can grab the first two columns of A and this set will have the requested properties.

$$S = \left\{ \begin{bmatrix} 1 \\ -4 \\ 1 \end{bmatrix},\ \begin{bmatrix} 2 \\ 1 \\ -7 \end{bmatrix} \right\}$$

(b) Construct S by a procedure that uses the row space of the transpose of A, $\mathcal{R}(A^t)$.

Solution: By Theorem CSRST the row space of A^t will equal the column space of A. We can obtain a set describing the row space of a matrix by row-reducing and keeping nonzero rows — this is Theorem BRS. These vectors will be linearly independent, and their span equals the subset, as requested.

$$\begin{bmatrix} 1 & -4 & 1 \\ 2 & 1 & -7 \\ 3 & -3 & -6 \\ 4 & -7 & -5 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & -3 \\ 0 & \boxed{1} & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Writing the rows of this matrix as column vectors, we have the set

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix},\ \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$$

3. Construct the extended echelon form of the matrix A below. For each of the four subsets, $\mathcal{R}(A)$, $\mathcal{N}(A)$, $\mathcal{C}(A)$ and $\mathcal{L}(A)$ (respectively: row space, null space, column space, left null space) find a linearly independent set of vectors whose span equals the desired subset by using information from the extended echelon form. (20 points)

$$A = \begin{bmatrix} 6 & -5 & 4 & 7 & -11 \\ -1 & 1 & -1 & -1 & 2 \\ -12 & 9 & -6 & -15 & 21 \\ 4 & -3 & 2 & 5 & -7 \end{bmatrix}$$

Solution: With four rows, we adjoin a 4×4 identity matrix and row-reduce to obtain the extended echelon form (Definition EEF),

$$\begin{bmatrix} 6 & -5 & 4 & 7 & -11 & 1 & 0 & 0 & 0 \\ -1 & 1 & -1 & -1 & 2 & 0 & 1 & 0 & 0 \\ -12 & 9 & -6 & -15 & 21 & 0 & 0 & 1 & 0 \\ 4 & -3 & 2 & 5 & -7 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & -1 & 2 & -1 & 0 & 3 & 0 & 1 \\ 0 & \boxed{1} & -2 & 1 & 1 & 0 & 4 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} & 2 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & 3 \end{bmatrix}$$

From this matrix we extract the submatrices C and L (as defined in Definition EEF),

$$C = \begin{bmatrix} \boxed{1} & 0 & -1 & 2 & -1 \\ 0 & \boxed{1} & -2 & 1 & 1 \end{bmatrix} \qquad L = \begin{bmatrix} \boxed{1} & 2 & 0 & -1 \\ 0 & 0 & \boxed{1} & 3 \end{bmatrix}$$

Applying Theorem FS, along with the indicated theorems about null spaces and row spaces, will give us the requested sets,

$$\mathcal{N}(A) = \mathcal{N}(C) = \left\langle \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} \right\rangle \qquad \text{Theorem BNS}$$

$$\mathcal{R}(A) = \mathcal{R}(C) = \left\langle \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \\ 1 \\ 1 \end{bmatrix} \right\} \right\rangle \qquad \text{Theorem BRS}$$

$$\mathcal{C}(A) = \mathcal{N}(L) = \left\langle \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -3 \\ 1 \end{bmatrix} \right\} \right\rangle \qquad \text{Theorem BNS}$$

$$\mathcal{L}(A) = \mathcal{R}(L) = \left\langle \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \end{bmatrix} \right\} \right\rangle \qquad \text{Theorem BRS}$$

4. Give an example of a 4×4 unitary matrix that is not the identity matrix. (15 points)

   Solution: the simplest route to such an examples is to begin with the 4×4 identity matrix and rearrange the columns. Since I_4 is unitary ($I_4^* I_4 = I_4 I_4 = I_4$), its columns form an orthonormal set by Theorem CUMOS. Any rearrangement of the columns will still be a matrix whose columns are an orthonormal set, and hence by Theorem CUMOS will be a unitary matrix. Here's a concrete example of such a solution, formed by swapping columns 1 and 2 of I_4.

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

5. Suppose that A is an $m \times n$ matrix, and B and C are $n \times p$ matrices. Give a careful proof that $A(B+C) = AB + AC$ (i.e. do not just simply quote a theorem from the book). (15 points)

   Solution: This is Theorem MMDAA. See the proof given there.

6. Suppose that A is a nonsingular matrix. Prove that $\left(A^t\right)^{-1} = \left(A^{-1}\right)^t$. (15 points)

   Solution: This is basically Theorem MIT. However the hypothesis is that A is nonsingular, so we need to first ascertain that A^{-1} really exists. Theorem NI does this for us. With A^{-1} in hand we can demonstrate that $\left(A^{-1}\right)^t$ functions as the inverse of A^t by meeting the requirements of Definition MI.