Math 290 Name: Key Dr. Beezer
Quiz VS Spring 2007

Show *all* of your work and *explain* your answers fully. There is a total of 90 possible points. If you use a calculator or Mathematica on a problem be sure to write down both the input and output.

1. Prove that the following subset of M_{22} is linearly independent. (15 points)

$$\left\{\begin{bmatrix}3&5\\2&8\end{bmatrix},\begin{bmatrix}3&5\\1&4\end{bmatrix},\begin{bmatrix}-1&-2\\1&7\end{bmatrix}\right\}$$

Solution: Begin with an arbitrary relation of linear dependence, Definition RLD,

$$a_1\begin{bmatrix}3&5\\2&8\end{bmatrix}+a_2\begin{bmatrix}3&5\\1&4\end{bmatrix}+a_3\begin{bmatrix}-1&-2\\1&7\end{bmatrix}=\mathbf{0}=\begin{bmatrix}0&0\\0&0\end{bmatrix}$$

The definitions of matrix addition (Definition MA), matrix scalar multiplication (Definition MSM), and matrix equality (Definition ME) yield a homogeneous system of four equations in the three unknowns, a_1, a_2 and a_3. The coefficient matrix of this system row-reduces as,

$$\begin{bmatrix}3&3&-1\\5&5&-2\\2&1&1\\8&4&7\end{bmatrix}\xrightarrow{\text{RREF}}\begin{bmatrix}\boxed{1}&0&0\\0&\boxed{1}&0\\0&0&\boxed{1}\\0&0&0\end{bmatrix}$$

this system has a unique solution, which must be the trivial solution (Theorem HSC). So there are no nontrivial relations of linear dependence, and thus the set ids linearly independent (Definition LI).

2. Prove that the set below is a spanning set of P_2, the vector space of polynomials of degree 2 or less. (15 points)

$$S=\left\{6+4x+5x^2,\,7+3x+2x^2,\,1+x+x^2\right\}$$

Solution: Let $a+bx+cx^2$ denote an arbitrary polynomial in P_2. To establish that S is a spanning set, we need to demonstrate that $q(x)$ can be expressed as a linear combination of the vectors in S (Definition TSS). Let α_1, α_2, α_3, denote scalars that might create such a linear combination,

$$\alpha_1\left(6+4x+5x^2\right)+\alpha_2\left(7+3x+2x^2\right)+\alpha_3\left(1+x+x^2\right)=a+bx+cx^2$$

Our definitions of the vector space operations and equality in P_2 (Example VCP) yield a sytem of three equations in the three unknowns α_1, α_2, α_3 having the augmented matrix,

$$\begin{bmatrix}6&7&1&a\\4&3&1&b\\5&2&1&c\end{bmatrix}$$

The coefficient matrix of this system is nonsingular, as you can see by checking that it row-reduces to the identity matrix, Theorem NMRRI. Thus, the system *always* has a solution by Theorem NMUS. So, even though we have not identified the exact form of a solution for the scalars α_1, α_2, α_3 we know they can be determined. So S is a spanning set for P_2.

Note that we could also prove that S is linearly independent, cite the dimension of P_2 as 3, and apply Goldilocks' Theorem (Theorem G) to get the spanning property of S.

3. Estabish, with proof, a spanning set for the vector space Y below. (15 points)

$$Y = \left\{ a + bx + cx^2 + dx^3 \in P_3 \,\middle|\, a - 3c + d = 0, b + 2c - 5d = 0 \right\}$$

Solution: We successively manipulate the expression for Y,

$$\begin{aligned}
Y &= \left\{ a + bx + cx^2 + dx^3 \in P_3 \,\middle|\, a - 3c + d = 0, b + 2c - 5d = 0 \right\} \\
&= \left\{ a + bx + cx^2 + dx^3 \in P_3 \,\middle|\, a = 3c - d, b = -2c + 5d \right\} \\
&= \left\{ (3c - d) + (-2c + 5d)\, x + cx^2 + dx^3 \,\middle|\, c, d \in \mathbb{C} \right\} \\
&= \left\{ \left(3c - 2cx + cx^2\right) + \left(-d + 5dx + dx^3\right) \,\middle|\, c, d \in \mathbb{C} \right\} \\
&= \left\{ c\left(3 - 2x + x^2\right) + d\left(-1 + 5x + x^3\right) \,\middle|\, c, d \in \mathbb{C} \right\} \\
&= \left\langle \left\{ 3 - 2x + x^2,\ -1 + 5x + x^3 \right\} \right\rangle
\end{aligned}$$

So we have established that $T = \left\{ 3 - 2x + x^2,\ -1 + 5x + x^3 \right\}$ is a spanning set for Y.

4. List the dimension of each vector space below. You only need to do as much work as you need to convince yourself of the answer, no other explanation is required (and therefore no partial credit will be given either). (15 points)

   (a) $M_{3,5}$, the vector space of all 3×5 matrices.

   Solution: By Theorem DM the simension is $3 \cdot 5 = 15$

   (b) In P_4, $W = \left\langle \left\{ 2, 4 + x, 9 + x + 5x^2 \right\} \right\rangle$.

   Solution: This is a linear independent set, thus a basis of W, and so $\dim(W) = 3$.

   (c) In M_{22}, $X = \left\{ \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}, \begin{bmatrix} 1 & -2 \\ 1 & -4 \end{bmatrix}, \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix} \right\}$.

   Solution: The first two matrices sum to the third. The first two are linearly independent, and so form a basis of X. Thus, $\dim(X) = 2$.

   (d) $Z = \left\{ \begin{bmatrix} 1 \\ 3 \\ -5 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \\ -10 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 5 \end{bmatrix} \right\}$.

   Solution: The matrix with these vectors as columns is singular. So $\dim(Z) \neq 3$. If Z had dimension 1, each vector would be a scalar multiple of any other. $\dim(Z) = 2$ is the only possibility left.

5. Illustrate the three-part test of Theorem TSS by proving that the set below is a subspace of $\mathbb{C}^4$. (15 points)

$$X = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \,\middle|\, 2a - b + c + 3d = 0 \right\}$$

Solution: See the solution to Exercise S.M20 for an example of how to construct this proof carefully.

6. Suppose that $S = \{\mathbf{v}_1,\, \mathbf{v}_2,\, \mathbf{v}_3,\, \ldots,\, \mathbf{v}_m\}$ is a linearly independent set of column vectors from $\mathbb{C}^n$ and A is a nonsingular matrix of size n. Prove that the set $T = \{A\mathbf{v}_1,\, A\mathbf{v}_2,\, A\mathbf{v}_3,\, \ldots,\, A\mathbf{v}_m\}$ is linearly independent. (15 points)

Solution: We begin with a relation of linear dependence on T,

$$\begin{aligned}
\mathbf{0} &= \alpha_1 A\mathbf{v}_1 + \alpha_2 A\mathbf{v}_2 + \alpha_3 A\mathbf{v}_3 + \cdots + \alpha_m A\mathbf{v}_m && \text{Definition RLD} \\
&= A\alpha_1 \mathbf{v}_1 + A\alpha_2 \mathbf{v}_2 + A\alpha_3 \mathbf{v}_3 + \cdots + A\alpha_m \mathbf{v}_m && \text{Theorem MMSMM} \\
&= A\left(\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \alpha_3 \mathbf{v}_3 + \cdots + \alpha_m \mathbf{v}_m\right) && \text{Theorem MMDAA}
\end{aligned}$$

Since A is assumed to be nonsingular (Definition NM), the vector in this final expression must be the zero vector,

$$\mathbf{0} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \alpha_3 \mathbf{v}_3 + \cdots + \alpha_m \mathbf{v}_m$$

Now, since S is linearly independent, Definition LI allows us to conclude that

$$\alpha_1 = \alpha_2 = \alpha_3 = \cdots = \alpha_m = 0$$

Now, by Definition LI this is exactly the conclusion we need to assert that T is linearly independent.