Show *all* of your work and *explain* your answers fully. There is a total of 100 possible points.

1. Compute the determinant of the matrix A without using a calculator. Is A nonsingular, and how do you know? (15 points)

$$A = \begin{bmatrix} 2 & -3 & 3 \\ 1 & -3 & -3 \\ 2 & 0 & 3 \end{bmatrix}$$

Solution: Expand about the second column, since it contains a zero entry (Theorem DEC). Expansion about the third row would be another good choice. You might also choose to use row operations to bring the matrix to an upper triangular form.

$$\begin{aligned} \det(A) &= \begin{vmatrix} 2 & -3 & 3 \\ 1 & -3 & -3 \\ 2 & 0 & 3 \end{vmatrix} \\ &= (-1)(-3)\begin{vmatrix} 1 & -3 \\ 2 & 3 \end{vmatrix} + (1)(-3)\begin{vmatrix} 2 & 3 \\ 2 & 3 \end{vmatrix} + (-1)(0)\begin{vmatrix} 2 & 3 \\ 1 & -3 \end{vmatrix} \\ &= 3(9) + (-3)(0) + 0 = 27 \end{aligned}$$

A nonzero determinant indicates a nonsingular matrix (Theorem SMZD).

2. Without using a calculator, find an eigenvector of the matrix B. (15 points)

$$B = \begin{bmatrix} 3 & 1 \\ -5 & 1 \end{bmatrix}$$

Solution: The characteristic polynomial (Definition CP) is

$$\begin{aligned} p_B(x) &= \det(B - xI_2) \\ &= \begin{vmatrix} 3-x & 1 \\ -5 & 1-x \end{vmatrix} \\ &= (3-x)(1-x) - (1)(-5) = 8 - 4x + x^2 = (x - (2+2i))(x - (2-2i)) \end{aligned}$$

So the eigenvalues of B are $\lambda = 2+2i, 2-2i$ (Theorem EMRCP). We will find an eigenvector for $\lambda = 2-2i$ as a nontrivial element of the null space of $B - (2-2i)I_2$ (Theorem EMNS),

$$B - (2-2i)I_2 = \begin{bmatrix} 3-(2-2i) & 1 \\ -5 & 1-(2-2i) \end{bmatrix} = \begin{bmatrix} 1+2i & 1 \\ -5 & -1+2i \end{bmatrix}$$

This matrix *must* be singular, and therefore the two columns must be scalar multiples of each other. With a 1 in the first entry of the second column, it is easy to construct a nontrivial relation of linear dependence on the columns, leading to the eigenvector,

$$\mathbf{x} = \begin{bmatrix} 1 \\ -(1+2i) \end{bmatrix} = \begin{bmatrix} 1 \\ -1-2i \end{bmatrix}$$

which you can check as an element of the null space, or as an eigenvector of B. By Theorem ERMCP, an eigenvector of $\lambda = 2+2i$ would be

$$\overline{\mathbf{x}} = \begin{bmatrix} 1 \\ -1+2i \end{bmatrix}$$

3. For each eigenvalue of the matrix C, compute the algebraic multiplicity, geometric multiplicity and eigenspace. Do not use a calculator for any aspect of this problem, except for row-reducing matrices. (40 points) (Hint: $\lambda = 2$ is one eigenvalue.)

$$C = \begin{bmatrix} 7 & 8 & -4 \\ -1 & 1 & 1 \\ 3 & 6 & 0 \end{bmatrix}$$

Solution: The characteristic polynomial of C can be computed as the determinant of $C - xI_3$, which will yield,

$$p_C(x) = 18 - 21x + 8x^2 - x^3 = (x-2)(-9+6x-x^2) = -(x-2)(x-3)^2$$

So the eigenvalues of C are $\lambda = 2$ and $\lambda = 3$. Algebraic multiplicities are

$$\alpha_C(2) = 1 \qquad\qquad \alpha_C(3) = 2$$

For eigenspaces, we form singular matrices and compute null spaces,

$$\lambda = 2 \qquad C - 2I_3 = \begin{bmatrix} 5 & 8 & -4 \\ -1 & -1 & 1 \\ 3 & 6 & -2 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & -\frac{4}{3} \\ 0 & \boxed{1} & \frac{1}{3} \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathcal{E}_C(2) = \mathcal{N}(C - 2I_3) = \left\langle \left\{ \begin{bmatrix} \frac{4}{3} \\ -\frac{1}{3} \\ 1 \end{bmatrix} \right\} \right\rangle = \left\langle \left\{ \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix} \right\} \right\rangle$$

$$\lambda = 3 \qquad C - 3I_3 = \begin{bmatrix} 4 & 8 & -4 \\ -1 & -2 & 1 \\ 3 & 6 & -3 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathcal{E}_C(3) = \mathcal{N}(C - 3I_3) = \left\langle \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right\} \right\rangle$$

From the bases of these eigenspaces we have the geometric multiplicities,

$$\gamma_C(2) = 1 \qquad\qquad \gamma_C(3) = 2$$

4. Suppose that A is a square matrix and $\mathbf{x}_1$ and $\mathbf{x}_2$ are eigenvectors of A for the eigenvalue λ, and α and β are any two complex numbers. Prove directly that $\alpha\mathbf{x}_1 + \beta\mathbf{x}_2$ is an eigenvector of A. (By "directly" I mean do not quote results about eigenspaces being subspaces.) (10 points)

   Solution:

$$\begin{aligned}
A\left(\alpha\mathbf{x}_1 + \beta\mathbf{x}_2\right) &= A\alpha\mathbf{x}_1 + A\beta\mathbf{x}_2 && \text{Theorem MMDAA}\\
&= \alpha A\mathbf{x}_1 + \beta A\mathbf{x}_2 && \text{Theorem MMSMM}\\
&= \alpha\lambda\mathbf{x}_1 + \beta\lambda\mathbf{x}_2 && \text{Definition EEM}\\
&= \lambda\alpha\mathbf{x}_1 + \lambda\beta\mathbf{x}_2 && \text{Property MCCN}\\
&= \lambda\left(\alpha\mathbf{x}_1 + \beta\mathbf{x}_2\right) && \text{Property DVAC}
\end{aligned}$$

   which establishes $\alpha\mathbf{x}_1 + \beta\mathbf{x}_2$ as an eigenvector of A (for the eigenvalue λ) (Definition EEM).

5. Suppose that A and B are similar matrices. Prove that A^2 and B^2 are similar. (10 points)

   Solution: See the solution to Acronym SD.T15.

6. Suppose that λ is an eigenvalue of a unitary matrix U. Prove that the modulus of λ is 1. (Recall that a matrix A is unitary if $A\left(\overline{A}\right)^t = I$ and the modulus of a complex number c is $\sqrt{c\overline{c}}$.) (10 points)

   Solution: Suppose $\mathbf{x}$ is an eigenvector of U for the eigenvalue λ. As such, $\mathbf{x} \neq \mathbf{0}$ and by Theorem PIP, $\langle\mathbf{x},\, \mathbf{x}\rangle \neq 0$. Then

$$\begin{aligned}
|\lambda| &= \sqrt{\lambda\overline{\lambda}} && \text{Definition MCN}\\
&= \sqrt{\frac{1}{\langle\mathbf{x},\, \mathbf{x}\rangle}\lambda\overline{\lambda}\left\langle\mathbf{x},\, \mathbf{x}\right\rangle} && \text{Property MICN}\\
&= \sqrt{\frac{1}{\langle\mathbf{x},\, \mathbf{x}\rangle}\left\langle\lambda\mathbf{x},\, \lambda\mathbf{x}\right\rangle} && \text{Theorem IPSM}\\
&= \sqrt{\frac{1}{\langle\mathbf{x},\, \mathbf{x}\rangle}\left\langle U\mathbf{x},\, U\mathbf{x}\right\rangle} && \text{Definition EEM}\\
&= \sqrt{\frac{1}{\langle\mathbf{x},\, \mathbf{x}\rangle}\left\langle\mathbf{x},\, \mathbf{x}\right\rangle} && \text{Theorem UMPIP}\\
&= \sqrt{1} = 1
\end{aligned}$$