Math 290 | Name: Key | Dr. Beezer
Quiz LT | | Spring 2007

Show *all* of your work and *explain* your answers fully. There is a total of 95 possible points.

1. Consider $T\colon \mathbb{C}^4 \mapsto \mathbb{C}^3$ defined below. (30 points)

$$T\left(\begin{bmatrix}a\\b\\c\\d\end{bmatrix}\right) = \begin{bmatrix}2a-3b+c+12d\\-a+b-c-5d\\2a+b+5c+4d\end{bmatrix}$$

(a) Find a basis for the kernel of T, $\mathcal{K}(T)$.

Solution: We desire elements of the domain that T takes to the zero vector of the codomain. this is the condition that,

$$T\left(\begin{bmatrix}a\\b\\c\\d\end{bmatrix}\right) = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

Replacing T by a generic element of the domain, and applying equality in $\mathbb{C}^3$, we arrive at a homogeneous system in the four variables a, b, c, d. The coefficient of this system row-reduces as,

$$\begin{bmatrix}2 & -3 & 1 & 12\\-1 & 1 & -1 & -5\\2 & 1 & 5 & 4\end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix}\boxed{1} & 0 & 2 & 3\\0 & \boxed{1} & 1 & -2\\0 & 0 & 0 & 0\end{bmatrix}$$

The null space of this coefficient matrix is the lernel of the linear transformation, so by an application of Theorem BNS we have

$$\mathcal{K}(T) = \left\langle\left\{\begin{bmatrix}-2\\-1\\1\\0\end{bmatrix}, \begin{bmatrix}-3\\2\\0\\1\end{bmatrix}\right\}\right\rangle$$

(b) Find a basis for the range of T, $\mathcal{R}(T)$.

Solution: Theorem SSRLT provides a spanning set for the range, simply by evaluating T at the elements of a spanning set of $\mathbb{C}^4$. Using the standard unit vectors (Definition SUV) we have the spanning set for $\mathcal{R}(T)$,

$$\{T(\mathbf{e}_1), T(\mathbf{e}_2), T(\mathbf{e}_3), T(\mathbf{e}_4)\} = \left\{\begin{bmatrix}2\\-1\\2\end{bmatrix}, \begin{bmatrix}-3\\1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\\5\end{bmatrix}, \begin{bmatrix}12\\-5\\4\end{bmatrix}\right\}$$

This set is not a basis, since it is not linearly independent (Theorem MVSLD). We will make the vectors the rows of a matrix, row-reduce, and keep the nonzero rows — a procedure based on theorems about the row space of a matrix,

$$\begin{bmatrix}2 & -1 & 2\\-3 & 1 & 1\\1 & -1 & 5\\12 & -5 & 4\end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix}\boxed{1} & 0 & -3\\0 & \boxed{1} & -8\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$$

Thus,

$$\mathcal{R}(T) = \left\langle \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -8 \end{bmatrix} \right\} \right\rangle$$

(c) Compute the pre-image of $\mathbf{v} = \begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix}$, $T^{-1}(\mathbf{v})$.

Solution: Theorem KPI tells us that we only need one element of the pre-image and the kernel of the linear transformation. The condion that $T(\mathbf{u}) = \begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix}$ leads to a system of three equations in four variables. We row-reduce the augmented matrix of this system,

$$\begin{bmatrix} 2 & -3 & 1 & 12 & -2 \\ -1 & 1 & -1 & -5 & 1 \\ 2 & 1 & 5 & 4 & -2 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 2 & 3 & -1 \\ 0 & \boxed{1} & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We really desire all of the solutions to this system, but with the kernel in-hand, we only need one solution. Setting both free variables to zero, we have $\mathbf{u} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ (note that columns 1 and 5 of the augmented matrix are negatives of each other). So

$$T^{-1}\left(\begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix}\right) = \mathbf{u} + \mathcal{K}(T) = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \left\langle \left\{ \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\} \right\rangle$$

2. Consider the linear transformation $R\colon P_2 \mapsto P_2$ defined below, where P_2 is the vector space of polynomials with degree at most 2. (30 points)

$$R\left(a + bx + cx^2\right) = (2a - b) + (2a - b + c)\, x + (a - b - c)\, x^2$$

(a) Prove that R is injective.

Solution: The condition that $R\left(a + bx + cx^2\right) = 0 + 0x + 0x^2$, together with our definition of polynomial equality leads to a homogeneous system of three equations in the three variables a, b, c. the coefficient matrix of this system is

$$\begin{bmatrix} 2 & -1 & 0 \\ 2 & -1 & 1 \\ 1 & -1 & -1 \end{bmatrix}$$

As this matrix row-reduces to the identity matrix, the only solution is trivial and $\mathcal{K}(R) = \left\{0 + 0x + 0x^2\right\}$. By Theorem KILT, R is injective.

(b) Prove that R is surjective.

Solution: We form a spanning set for the range by evaluating R on the basis $\left\{1,\, x,\, x^2\right\}$ of P_2 and apply Theorem SSRLT,

$$\mathcal{R}(R) = \left\langle \left\{ R\,(1)\,,\, R\,(x)\,,\, R\left(x^2\right) \right\} \right\rangle = \left\langle \left\{ 2 + 2x + x^2,\, -1 - x - x^2,\, x - x^2 \right\} \right\rangle$$

Check that this set is linearly independent, so $\dim \mathcal{R}(R) = 3 = \dim P_2$. By Theorem EDYES, $\mathcal{R}(R) = P_2$, so R is surjective by Theorem RSLT.

(c) Since R is both surjective and injective, it is invertible. The linear transformation S below is the inverse of R. Demonstrate the use of S in the definition of an invertible linear transformation, by working one half of the necessary verifications.

$$S\left(a + bx + cx^2\right) = (2a - b - c) + (3a - 2b - 2c)\, x + (-a + b)\, x^2$$

Solution: This question asks you to check that

$$\begin{aligned} R \circ S\left(p + qx + rx^2\right) &= R\left(S\left(p + qx + rx^2\right)\right) \\ &= R\left((2p - q - r) + (3p - 2q - 2r)\, x + (-p + q)\, x^2\right) \\ &= (2\,(2p - q - r) - (3p - 2q - 2r)) + \\ &\qquad (2\,(2p - q - r) - (3p - 2q - 2r) + (-p + q))\, x + \\ &\qquad ((2p - q - r) - (3p - 2q - 2r) - (-p + q))\, x^2 \\ &= p + qx + rx^2 \\ &= I_{P_2}\left(p + qx + rx^2\right) \end{aligned}$$

Similarly, you could show that $S \circ R = I_{P_2}$.

3. For the linear transformation T in Problem 1, find two elements of the domain, $\mathbf{x}$ and $\mathbf{y}$, neither the zero vector, and such that $T(\mathbf{x}) = T(\mathbf{y})$. (10 points)

Solution: The requested condition is true exactly when $\mathbf{x} - \mathbf{y}$ is an element of the kernel of T, as we see in,

$$T(\mathbf{x} - \mathbf{y}) = T(\mathbf{x}) - T(\mathbf{y}) = \mathbf{0}$$

There are infinitely many solutions. We'll choose any nonzero vector for $\mathbf{x}$, say $\mathbf{x} = \begin{bmatrix}1\\2\\3\\4\end{bmatrix}$. To form $\mathbf{y}$, we simply add any element of the kernel of T. We will choose the sum of the two basis vectors in our solution to Problem 1a,

$$\mathbf{y} = \begin{bmatrix}1\\2\\3\\4\end{bmatrix} + \left(\begin{bmatrix}-2\\-1\\1\\0\end{bmatrix} + \begin{bmatrix}-3\\2\\0\\1\end{bmatrix}\right) = \begin{bmatrix}-4\\3\\4\\5\end{bmatrix}$$

By our general work above, we know $\mathbf{x}$ and $\mathbf{y}$ have the desired property, however, we could verify that

$$T(\mathbf{x}) = T(\mathbf{y}) = \begin{bmatrix}47\\-22\\35\end{bmatrix}$$

4. For the linear transformation T in Problem 1, find an element of the codomain, $\mathbf{w}$, such that there is no element $\mathbf{x}$ with $T(\mathbf{x}) = \mathbf{w}$. (10 points)

Solution: This question asks for a vector in $\mathbb{C}^3$ that is not in the range of T. Suppose we think about a vector $\mathbf{v}$ that *is* an element of the range of T and its first two entries are both 1. Then from our expression for the range of T in the solution to Problem 1b,

$$\mathcal{R}(T) = \left\langle\left\{\begin{bmatrix}1\\0\\-3\end{bmatrix}, \begin{bmatrix}0\\1\\-8\end{bmatrix}\right\}\right\rangle$$

we see that our hypothetical $\mathbf{v}$ in the range can only be $\begin{bmatrix}1\\1\\-11\end{bmatrix}$. Any other vector with its first two entries equal to 1 *must* lie outside of the range. So, for example, we could choose $\mathbf{w} = \begin{bmatrix}1\\1\\632\end{bmatrix}$ to obtain a vector of $\mathbb{C}^3$ outside of the range. To check our answer, we could try to find a solution for $\mathbf{x}$ in $T(\mathbf{x}) = \begin{bmatrix}1\\1\\632\end{bmatrix}$, but eventually we would reach an inconsistent system of equations.

5. Suppose that $Q\colon U \mapsto V$ is an injective linear transformation. Let $C = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_m\}$ be a linearly independent set of vectors in U. Prove that $\{Q(\mathbf{u}_1), Q(\mathbf{u}_2), Q(\mathbf{u}_3), \ldots, Q(\mathbf{u}_m)\}$ is a linearly independent set in V. (Do not quote a similar theorem about a basis as your proof.) (15 points)

Solution: This is Theorem ILTLI. See the proof there.