Math 290 Name: Key Dr. Beezer
Quiz R Spring 2007

Show *all* of your work and *explain* your answers fully. There is a total of 125 possible points. Objects are defined on the last page, you might find it convenient to seperate that page from the rest of the exam.

1. Find the vector representation of $\mathbf{x}$, relative to the basis B, $\rho_B(\mathbf{x})$. (10 points)

   Solution: Express $\mathbf{x}$ as a linear combination of the elements of the basis B,

$$\begin{bmatrix} -1 & 3 \\ 3 & -2 \end{bmatrix} = (-1)\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + (-2)\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + 3\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

   So

$$\rho_B(\mathbf{x}) = \begin{bmatrix} -1 \\ -2 \\ 3 \end{bmatrix}$$

2. Find the vector representation of $\mathbf{x}$, relative to the basis D, $\rho_D(\mathbf{x})$. (10 points)

   Solution: Express $\mathbf{x}$ as a linear combination of the elements of the basis D. A linear combination of the basis elements, with unknown scalars, would lead to a system of three equations in three unknowns, and a nonsingular coefficient matrix. From the unique solution, we construct the matrix equality

$$\begin{bmatrix} -1 & 3 \\ 3 & -2 \end{bmatrix} = (-21)\begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} + (-20)\begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix} + 8\begin{bmatrix} 0 & 3 \\ 3 & 2 \end{bmatrix}$$

   So

$$\rho_D(\mathbf{x}) = \begin{bmatrix} -21 \\ -20 \\ 8 \end{bmatrix}$$

3. Use a change-of-basis matrix to convert $\rho_D(\mathbf{x})$ into $\rho_B(\mathbf{x})$. (15 points)

   Solution: The question asks us to use $C_{D,B}$. This matrix is constructed by taking the basis vectors of D and writing them as linear combinations of the basis vectors in B,

$$\rho_B\left(\begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}\right) = \rho_B\left(1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + (-2)\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + 1\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$\rho_B\left(\begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}\right) = \rho_B\left((-1)\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 3\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + 0\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}$$

$$\rho_B\left(\begin{bmatrix} 0 & 3 \\ 3 & 2 \end{bmatrix}\right) = \rho_B\left(0\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 2\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + 3\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$$

   These column vectors are the columns of the change-of-basis matrix,

$$C_{D,C} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 2 \\ 1 & 0 & 3 \end{bmatrix}$$

The conversion is

$$\rho_B(\mathbf{x}) = C_{D,B}\rho_D(\mathbf{x}) = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 2 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} -21 \\ -20 \\ 8 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ 3 \end{bmatrix}$$

4. Build the matrix representation of T relative to B and C, $M^T_{B,C}$. (10 points)

   Solution: We evaluate T at the basis elements of B and coordinatize them relative to C,

$$\rho_C\left(T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right)\right) = \rho_C(2+x) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$
$$\rho_C\left(T\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right)\right) = \rho_C(5+2x) = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$
$$\rho_C\left(T\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right)\right) = \rho_C(4+3x) = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

   These column vectors are the columns of the matrix representation (Definition MR),

$$M^T_{B,C} = \begin{bmatrix} 2 & 5 & 4 \\ 1 & 2 & 3 \end{bmatrix}$$

5. Build the matrix representation of T relative to D and E, $M^T_{D,E}$. (10 points)

   Solution: We evaluate T at the basis elements of D and coordinatize them relative to E,

$$\rho_E\left(T\left(\begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}\right)\right) = \rho_E(-4) = \rho_E((-16)(1+x) + 4(3+4x)) = \begin{bmatrix} -16 \\ 4 \end{bmatrix}$$
$$\rho_E\left(T\left(\begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}\right)\right) = \rho_E(13+5x) = \rho_E(37(1+x) + (-8)(3+4x)) = \begin{bmatrix} 37 \\ -8 \end{bmatrix}$$
$$\rho_E\left(T\left(\begin{bmatrix} 0 & 3 \\ 3 & 2 \end{bmatrix}\right)\right) = \rho_E(22+13x) = \rho_E(49(1+x) + (-9)(3+4x)) = \begin{bmatrix} 49 \\ -9 \end{bmatrix}$$

   These column vectors are the columns of the matrix representation (Definition MR),

$$M^T_{D,E} = \begin{bmatrix} -16 & 37 & 49 \\ 4 & -8 & -9 \end{bmatrix}$$

   Notice that we could find this matrix directly with two change-of-basis matrices and an application of Theorem MRCB.

6. Use $M^T_{B,C}$ and $\mathbf{x}$ to illustrate the Fundamental Theorem of Matrix Representations (Theorem FTMR). (10 points)

   Solution: On the one hand,

$$T(\mathbf{x}) = T\left(\begin{bmatrix} -1 & 3 \\ 3 & -2 \end{bmatrix}\right) = 4x$$

   Now, more slowly, employing Theorem FTMR,

$$\begin{aligned} T(\mathbf{x}) &= \rho_C^{-1}\left(M^T_{B,C}\rho_B(\mathbf{x})\right) \\ &= \rho_C^{-1}\left(\begin{bmatrix} 2 & 5 & 4 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} -1 \\ -2 \\ 3 \end{bmatrix}\right) \\ &= \rho_C^{-1}\left(\begin{bmatrix} 0 \\ 4 \end{bmatrix}\right) \\ &= 0(1) + 4(x) = 4x \end{aligned}$$

7. Use $M^T_{D,E}$ and $\mathbf{x}$ to illustrate the Fundamental Theorem of Matrix Representations (Theorem FTMR). Comment on the relationship between this problem and the previous one. (15 points)

   Solution: Just as before,

$$T(\mathbf{x}) = T\left(\begin{bmatrix} -1 & 3 \\ 3 & -2 \end{bmatrix}\right) = 4x$$

   Now, more slowly, employing Theorem FTMR,

$$\begin{aligned} T(\mathbf{x}) &= \rho_E^{-1}\left(M^T_{D,E}\rho_D(\mathbf{x})\right) \\ &= \rho_E^{-1}\left(\begin{bmatrix} -16 & 37 & 49 \\ 4 & -8 & -9 \end{bmatrix}\begin{bmatrix} -21 \\ -20 \\ 8 \end{bmatrix}\right) \\ &= \rho_E^{-1}\left(\begin{bmatrix} -12 \\ 4 \end{bmatrix}\right) \\ &= (-12)(1+x) + 4(3+4x) = 4x \end{aligned}$$

   In both cases we get the "right" value for the linear transformation, and this value does not change in the two problems. It is just the intermediate representations that vary as we work with different pairs of bases.

8. Prove that the linear transformation R is invertible. (10 points)

   Solution: Build a matrix representation of R, so that we can analyze it with tools for analyzing matrices. We are given the freedom here to choose any basis (bases?) we like for P_1, lets use an easy one, say $G = \{1,\, x\}$. Then the matrix representation of R relative to G is, according to Definition MR,

$$M^{R}_{G,G} = \begin{bmatrix} 3 & -4 \\ 2 & -3 \end{bmatrix}$$

   The matrix representation has determinant -1, which is nonzero, so by Theorem SMZD the matrix representation is an invertible matrix. This means that R is an invertible linear transformation.

9. Find the inverse of R, R^{-1}. (15 points)

   Solution: Applying Theorem FTMR and Theorem IMR,

$$\begin{aligned} R^{-1}(a+bx) &= \rho_G^{-1}\left(M^{R^{-1}}_{G,G}\,\rho_G(a+bx)\right) \\ &= \rho_G^{-1}\left(\left(M^{R}_{G,G}\right)^{-1}\begin{bmatrix} a \\ b \end{bmatrix}\right) \\ &= \rho_G^{-1}\left(\begin{bmatrix} 3 & -4 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}\right) \\ &= \rho_G^{-1}\left(\begin{bmatrix} 3a-4b \\ 2a-3b \end{bmatrix}\right) \\ &= (3a-4b)+(2a-3b)\,x \end{aligned}$$

   Which, only coincidentally, is identical to R.

10. Find a basis, F, for P_1 such that the matrix representation of R relative to F, $M^{R}_{F,F}$, is a diagonal matrix. (20 points)

    Solution: We desire eigenvectors of the linear transformation, which can be found from the eigenvectors of any matrix representation, suitably un-coordinatized. The eigenvalues and eigenvectors of $M^{R}_{G,G}$ are,

$$\lambda = -1 \qquad \mathcal{E}_{M^{R}_{G,G}}(-1) = \left\langle\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}\right\rangle$$

$$\lambda = 1 \qquad \mathcal{E}_{M^{R}_{G,G}}(1) = \left\langle\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}\right\rangle$$

    Uncoordinatizing these eigenvectors, relative to G, forms the basis elements we desire,

$$\rho_G^{-1}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = 1+x \qquad \rho_G^{-1}\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = 2+x$$

$$
\begin{aligned}
S_2 &= \left\{ A \in M_{22} \mid A = A^t \right\} \\
&= \text{vector space of all } 2 \times 2 \text{ symmetric matrices}
\end{aligned}
$$

NOTE: S_2 has dimension 3 (not 4), as the two sets below are bases

$$B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$$

$$D = \left\{ \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 3 \\ 3 & 2 \end{bmatrix} \right\}$$

$$\mathbf{x} = \begin{bmatrix} -1 & 3 \\ 3 & -2 \end{bmatrix} \in S_2$$

$$
\begin{aligned}
P_1 &= \{ a + bx \mid a, b \in \mathbb{C} \} \\
&= \text{vector space of all polynomials in } x \text{ with degree 1 or less}
\end{aligned}
$$

$$C = \{1,\, x\}$$

$$E = \{1 + x,\, 3 + 4x\}$$

$$\mathbf{y} = 5 - 3x \in P_1$$

$$T\colon S_2 \mapsto P_1, \quad T\left(\begin{bmatrix} a & b \\ b & c \end{bmatrix} \right) = (2a + 4b + 5c) + (a + 3b + 2c)\, x$$

$$R\colon P_1 \mapsto P_1, \quad R\,(a + bx) = (3a - 4b) + (2a - 3b)\, x$$