Distance Regular Graphs
Simply Explained

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Adjacency Algebra and Distance Regular Graphs:

In this paper, we will discover some interesting properties of a particular kind of graph, called distance regular graphs, using algebraic graph theory. We will begin by developing some definitions that will allow us to explore the relationship between powers of the adjacency matrix of a graph and its eigenvalues, and ultimately give us particular insight into the eigenvalues of distance regular graphs.

Basis for the Polynomial of an Adjacency Matrix

We all know that for a given graph G, the powers of its adjacency matrix, A, have as their entries the number of n-walks between two vertices. More succinctly, we know that $[A_n]_{i,j}$ is the number of n-walks between the two vertices v_i and v_j. We also know that for a graph with diameter d, the first d powers of A are all linearly independent.

Now, let us think of the set of all polynomials of the adjacency matrix, A, for a graph G. We can think of any member of the set as a linear combination of powers of A. If the dimensoin of this set is finite, we should be able to find some basis for the set, but we're already getting ahead of ourselves. Lets introduce a few definitions so we can more easily work with this idea:

Definition AA
Adjacency Algebra
The adjacency algebra of a graph G is the set of all polynomials of the adjacency matrix A for G. We will refer to the adjacency algebra of a graph G, by $\mathcal{A}(G)$.

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While this set may seem a little difficult to get a handle on (or just pointless anyway), we actually already have information on the minimum size of this basis:

Theorem MDAAG
Minimum Dimension of the Adjacency Algebra of a Graph
For a graph G with diameter d, the minimum dimension of $\mathcal{A}(G)$, is $d + 1$: $\dim(\mathcal{A}(G)) \geq d + 1$.

$\square$

Proof   We know that the first d powers of A are all linearly independent, including A^0, meaning $\{A^0, A^1, \ldots, A^d\}$ is a linearly independent set, with size $d + 1$. This means that no smaller set could span $\mathcal{A}(G)$, but we do not know if a larger set will still be linearly independent.

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Regular Graphs

We quickly narrow our field of view to regular graphs. Regular graphs are an interesting subset of graphs, where every vertex of a regular graph has the same number of adjacent vertices.

Definition KRG
k-**regular graphs**
A graph is k-regular if all vertices of the graph G have the same degree, k:
$deg(v_i) = k$ for all $1 < i < n$.

For any k-regular graph, G, and the all-one $n \times n$ matrix J, where $[J]_{i,j} = 1$ for all $1 \le i \le n$ $1 \le j \le$, we can show that the product of the $n \times n$ adjacency matrix, A, with J will be kJ.

Theorem AMAOV
Adjacency Matrices and All-One Vectors
Given the $n \times n$ adjacency matrix, A, for a k-regular graph, and an $n \times n$ matrix J, where $[J]_{i,j} = 1$ for all i, $1 < i < n$ and all j, $1 < j < n$,

$$AJ = kJ.$$

□

Proof First we will remember that for a k-regular graph, G, with the adjacency matrix A, k is an eigenvector of A, and the all-one vector is in the eigenspace of k:

$$\vec{j} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}.$$

For this proof we will appeal to the definition of matrix multiplication where the ith column of the product is the product of the first matrix with the column i of the other matrix:

$$\begin{aligned} AJ &= [AJ_1 | AJ_2 | \cdots | AJ_n] \\ &= [A\vec{j} | A\vec{j} | \cdots | A\vec{j}] \\ &= [k\vec{j} | k\vec{j} | \cdots | k\vec{j}] \\ &= k[\vec{j} | \vec{j} | \cdots | \vec{j}] \\ &= kJ. \end{aligned}$$

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While this theorem may not seem particularly interesting for the moment, it will become very handy later in our discussion of distance-regular graphs.

Distance Regular Graphs

Distance regular graphs are a subset of regular graphs. Distance regular graphs have an *intersection array* $\{b_0, b_1 \ldots, b_{d-1}; c_1, c_2, \ldots, c_d\}$ where for any two vertices v and u that are r distance apart, $\delta(v, u) = r$. b_r is the number of verteces that are adjacent to u and at a distance $r + 1$ to v. c_r is the number of vertices that are adjacent to u and a distance of $r - 1$ from v. These numbers will depend only on the distance r and will hold for any pair of (v, u).

Another way to describe these graphs would be to define a subgraph of G, $G_r(v_o)$, such that $G_r(v_o)$ contains all the vertices that are r distance from v_o or less, and describe distance regular graphs and intersection numbers in the following way:

Definition RDG
Distance Regular Graphs
A distance regular graph, G is a regular graph of degree k, and diameter d, with an array (of non-zero elements) $\{b_o, b_1, \ldots, b_{d-1}; c_1, c_2, \ldots, c_d\}$, called the *intersection array*, such that for each pair of vertices v, u with distance r:
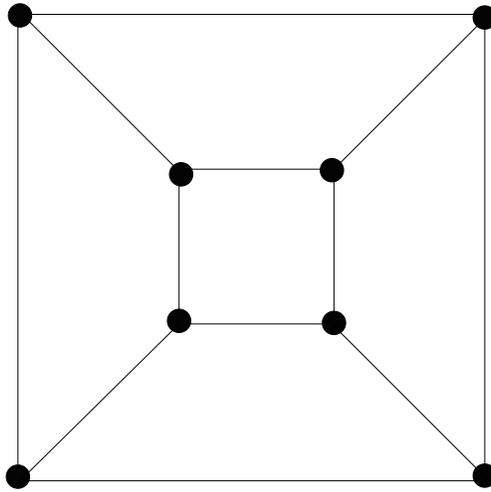
where $b_0 = k$, $c_1 = 1$, and

(1)   the number of vertices in $G_{r-1}(v)$ that are adjacent to u is c_r $(1 \leq r \leq d)$,

(2)   the number of verities in $G_{r+1}(v)$ that are adjacent to u is b_r $(0 \leq r \leq d-1)$.

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Example CG
Cubical Graph

Wolfram Research Institute, "Cubical Graph", mathworld.wolfram.com/CubicalGraph

This graph is called a cubical graph, because it can be considered a graphical representation of a cube, where the verticies are corners, and the edges are edges of the cube. It is distance regular with degree $k = 3$, distance 3, and an intersection array:

$$\{3, 3, 3; 1, 2, 3\}$$

$\diamond$

Adjacency Algebra for Distance-Regular Graphs

Instead of examining the powers of the adjacency matrix for distance-regular graphs, we are going to define a set of $n \times n$ distance matrices $\{A_0, A_1, \ldots, A_d\}$ such that:

$$[A_r]_{i,j} = \begin{cases} 1 & \delta(v_i, v_j) = r \\ 0 & \delta(v_i, v_j) \neq r \end{cases}$$

We see that $A_0 = I$ is just the identity matrix, and that $A_1 = A$, is simply our standard adjacency matrix. Here is an example of a set of distance matrices extended from our previous example:

Example DMCG
Distance Matrices for the Cubical Graph
A cubical graph has diameter 3, and 8 verticies, so there will be four 8×8 distance graphs, $\{A_0, A_1, A_2, A_3\}$:

$$A_0 = I_8$$

$$A_1 = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \quad A_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\diamond$

Now, you might be wondering why we would want to make up this new set of matrices when we might already have a good (or at least better) understanding of the powers of the adjacency matrix. The payoff will become apparent after these next two theorems. The first theorem is just set up for the result of the second one.

Theorem EPDV
Entries of the Product Correspond to the Distance Between the Entries
For any distance regular graph G, with adjacency matrix A, and the set of distance matrices $\{A_0, A_1, \ldots, A_d\}$, The entries of the product of AA_r for $0 < r < d$, will correspond to the number of vertices w of G, where the distance between v_i and w is 1, and the distance between v_j and w is r.

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Proof

$$[AA_r]_{i,j} = \sum_{w=1}^{n} [A]_{i,w}[A_r]_{w,j}$$

$$= \sum_{\delta(w,v_i)=1} [A]_{i,w}[A_r]_{w,j} + \sum_{\delta(w,v_i)\neq 1} [A]_{iw}[A_r]_{w,j}$$

$$= \sum_{\delta(w,v_i)=1} (1)[A_r]_{w,j} + \sum_{\delta(w,v_i)\neq 1} (0)[A_r]_{w,j}$$

$$= \sum_{\substack{\delta(w,v_1)=1 \\ \delta(w,v_j)=r}} (1)[A_r]_{w,j} + \sum_{\substack{\delta(w,v_1)=1 \\ \delta(w,v_j)\neq r}} (1)[A_r]_{w,j}$$

$$= \sum_{\substack{\delta(w,v_1)=1 \\ \delta(w,v_j)=r}} (1)(1) + \sum_{\substack{\delta(w,v_1)=1 \\ \delta(w,v_j)\neq r}} (1)(0)$$

$$= \sum_{\substack{\delta(w,v_1)=1 \\ \delta(w,v_j)=r}} 1$$

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Theorem PALC
The Product as a Linear Combination
The product AA_r can be written as a linear combination of A_{r-1}, A_r, and A_{r+1}, such that:

$$AA_r = b_{r-1}A_{r-1} + a_r A_r + c_{r+1}A_{r+1}.$$

Where b_r and c_r are the intersection numbers, and $a_r = k - b_r - c_r$.

□

Proof We know

$$[AA_r]_{i,j} = \sum_{\substack{\delta(w,v_1)=1 \\ \delta(w,v_j)=r}} 1,$$

and we see that since w is adjacent to v_i and a distance of r from v_j, then $\delta(v_i, v_j)$,is either $r-1$, $r+1$, or r. We know that the number of vertices that are a distance $r-1$ from v_j, and adjacent to v_i is b_{r-1}, that the number of vertices that are $r+1$ away from v_j is c_{r+1}, and since there are only k vertices adjacent to v_i, then there are only $k - b_{r-1} - c_{r+1}$ vertices that are r distance from v_j and adjacent to v_i.

$$[AA_r]_{i,j} = b_{r-1}[A_{r-1}]_{i,j} + a_r[A_i]_{i,j} + c_{r+1}[A_{i+1}]_{i,j}$$

for all i, $1 \le i \le n$ and all j, $1 \le j \le n$, and

$$AA_r = b_{r-1}A_{r-1} + a_r A_i + c_{r+1}A_{i+1}$$

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Ok, that last proof may have been interesting, but the really exciting part comes up next. We can now use induction to prove that we can write any A_r as a polynomial of just A.

Theorem DMAP
Distance Matrices as an Adjacency Polynomial
Given a distance regular graph, G, with degree k, diameter d, and a set of distance matrices $\{A_0, A_1, \ldots, A_d\}$, then A_r, $0 \le r \le d$, can be written as a polynomial of A, with degree r:

$$A_r = P_r(A)$$

□

Proof Let us note that

$$A_0 = I$$
$$= A^0$$

and,

$$A_1 = A.$$

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So we only need to look at the cases where $r > 1$. We will start by rewriting our equality from Theorem a little differently:

$$AA_r = b_{r-1}A_{r-1} + a_r A_r + c_{r+1}A_{r+1}$$
$$c_{r+1}A_{r+1} = AA_r - b_{r-1}A_{r-1}ar A_r,$$

and let us change our indices so that,

$$c_r A_r = AA_{r-1} - b_{r-2}A_{r-2}a_{r-1}A_{r-1}.$$

For our induction, we will use $r = 2$ as our base case:

$$\begin{aligned} c_2 A_2 &= AA_1 - b_0 A_0 - a_1 A_1 \\ &= AA - b_0 I - a_1 A \\ &= A^2 - a_1 A - b_0 I. \end{aligned}$$

So we see that A_2 can be written as a polynomial in A, and that the polynomial has degree 2. Now we look at the rth case, where $r \leq d$:

$$c_r Ar = AA_{r-1} - b_{r-2}A_{r-2}a_{r-1}A_{r-1}.$$

Our induction hypothesis says that A_{r-1} and A_{r-2} can both be written as polynomials of degree $r - 1$, and $r - 2$, respectively, so:

$$\begin{aligned} c_r A_r &= AP_{r-1}(A) - b_{r-2}P_{r-2}(A) - a_{r-1}P_{r-1}(A) \\ &= P_r(A) - b_{r-2}P_{r-2}(A) - a_{r-1}P_{r-1}(A) \\ &= P_r(A) \\ A_r &= \frac{1}{c_r} P_r(A) \\ &= P_r(A). \end{aligned}$$

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Ok, that was fun and enlightening. Now we know that the distance matrices of a distance-regular graph are really just linear combinations of powers of the adjacency matrix of that graph! We now have a way to relate the distance matrices of G to the adjacency algebra for G; lets see where this leads us.

Theorem DDRG
Dimension of a Distance Regular Graph
For a distance-regular graph, G, with diameter d, the dimension of the adjacency algebra of G, $\mathcal{A}(G)$, is

$$\dim(\mathcal{A}(G)) = d + 1,$$

and the set of distance matrices $\{A_0, A_1, \ldots, A_d\}$ is a basis for $\mathcal{A}(G)$.

□

Proof First, let us notice that because the graph is connected, and distance regular, the sum of all the distance matrices will equal the all-1 $n \times n$ matrix, J. Also, let us remind ourselves that for a regular graph of degree k, $AJ = kJ$.

$$\begin{aligned} (A - kI)(A_0 + A_1 + A_2 + \ldots + A_d) &= (A - kI)(A_0 + A_1 + A_2 + \ldots + A_d) \\ &= (A - kI)(J) \\ &= (AJ - kIJ) \\ &= kJ - kJ \\ &= \mathcal{O}. \end{aligned}$$

We know that the sum of all the distance matrices of G can be written as a polynomial in A that will be degree d, then we see that $(A - kI)(A_0 + A_1 + A_2 + \ldots + A_d)$ is a polynomial of A with degree $d + 1$. Since this non-trival linear combination of $d + 1$ powers of A equals the zero matrix, $\mathcal{O}$, we know that the first $d + 1$ powers of A are linearly dependent. Therefore $\dim(\mathcal{A}(G)) \leq d + 1$. However, we also know that the dimension of the adjacency algebra of G must be at least $d + 1$, $\dim(\mathcal{A}(G)) \geq d + 1$ (Note that the set $\{A^0, A^1, \ldots, A^d\}$ has $d + 1$ elements in it, and it spans the polynomial $P_d(A)$). Therefore

$$d + 1 \leq \dim(\mathcal{A}(G)) \leq d + 1$$
$$\dim(\mathcal{A}(G)) = d + 1.$$

It also follows that the set $\{A_0, A_1, \ldots, A_d\}$ is linearly independent because no two vectors v_1, v_j can have two different distances from each other, so for any position $\{i, j\}$ in the set of distance matrices, there is only one matrix with a one entry in that position, and all the other matrices have zero. Since this set is a linearly independent set of $d + 1$ elements, it must span $P(A)$, and be a basis for $\mathcal{A}(G)$.

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Phew, that feels good, doesn't it? We can rest easy knowing that we can easily find the dimension of $\mathcal{A}(G)$ for any distance-regular graph, G. Even better, we can also find a wonderful basis for $\mathcal{A}(G)$ without having to compute powers of A! How could it get any better?

Theorem EVDRG (formerly known as Theorem IGB)
Eigenvalues of Distance Regular Graphs
(formerly known as Theorem It Gets Better)
For any distance-regular graph, G, with diamter d, there exist $d + 1$ distinct eigenvalues.

□

Proof For this proof we will empoly the minimal polynomial of A. The minimal polynomial is the smallest degree non-trivial polynomial such that $P(A) = 0$. We know from the Cayley-Hamilton theorem that this polynomial will always divide the characteristic polynomial of A, and so the roots of the minimal polynomial are all the eigenvalues of A. Since we know that the first d powers of A are linearly independent, we know that there is no polynomial of degree d or less, such that $P_d(A) = 0$, meaning that the minimal polynomial must be at least degree $d + 1$. A $d + 1$ polynomial will have $d + 1$ roots, therefore, A has $d + 1$ eigenvalues.

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While we know that the $\dim(\mathcal{A}(G))$ of any graph must be at least $d + 1$, we have found that for a distance-regular graph, the dimension equals exactly $d + 1$. In addition, we have found two bases for this space, but each is notably distinct, because we found our set of $d + 1$ powers of A using primarily linear algebra, but we could find our set of distance matrices simply by looking at the graph. Imagine, we could find a basis for the entire set of polynomials of A without ever taking a power of anything, let alone a matrix! Also, although we know that any graph must have at least $d + 1$ eigenvalues, again we find that distance-regular graphs have only this minimum number. So, just by realizing a graph is distance-regular, we automatically know how many eigenvalues it's adjacency matrix has! This should feel like a remarkable relationship between two areas of mathematics.

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