Math 290 Name: Key Dr. Beezer
Quiz V Spring 2008

Show *all* of your work and *explain* your answers fully. There is a total of 90 possible points. If you use a calculator or software package on a problem be sure to write down both the input and output.

1. Determine if the set R below is linearly independent. (15 points)

$$R = \left\{ \begin{bmatrix} 1 \\ 3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} -4 \\ 3 \\ 4 \\ -5 \end{bmatrix} \right\}$$

Solution: According to Theorem LIVRN, we can answer this question quickly through an analysis of a matrix whose columns are the vectors of R. Specifically, we need the reduced row-echelon form of this matrix,

$$\begin{bmatrix} 1 & 2 & -4 \\ 3 & 1 & 3 \\ -1 & -2 & 4 \\ 2 & 3 & -5 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 2 \\ 0 & \boxed{1} & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

We have $n = 3$ vectors in R and from the matrix in reduced row-echelon form, we find $r = 2$ nonzero rows. Since $r < n$, Theorem LIVRN implies that R is linearly dependent.

2. Either $\mathbf{x}$ or $\mathbf{y}$ is an element of $\langle Q \rangle$ (but not both). Determine which vector is in $\langle Q \rangle$ and provide convincing evidence for membership in $\langle Q \rangle$. (15 points)

$$Q = \left\{ \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 9 \\ -5 \end{bmatrix} \right\} \qquad \mathbf{x} = \begin{bmatrix} 3 \\ 6 \\ -3 \end{bmatrix} \qquad \mathbf{y} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$$

Solution: According to Definition SSCV we ask if there are scalars a_1, a_2, a_3 so that

$$a_1 \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} + a_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + a_3 \begin{bmatrix} 4 \\ 9 \\ -5 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ -3 \end{bmatrix}$$

By Theorem SLSLC, such scalars will be a solution to the system whose augmented matrix row-reduces as follows,

$$\begin{bmatrix} 2 & 1 & 4 & 3 \\ 3 & 0 & 9 & 6 \\ -1 & 1 & -5 & -3 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 3 & 2 \\ 0 & \boxed{1} & -2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

By Theorem RCLS, this system is consistent, so we know there is a solution. a_3 is a free variable, so we can set $a_3 = 0$, and determine that $a_1 = 2$, $a_2 = -1$ is the remainder of a solution. So we can write $\mathbf{x}$ as a linear combination of the vectors in Q, and this *alone* would be enough evidence to conclude that $\mathbf{x} \in \langle Q \rangle$.

$$(2) \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} + (-1) \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + (0) \begin{bmatrix} 4 \\ 9 \\ -5 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ -3 \end{bmatrix}$$

A similar analysis for $\mathbf{y}$ would lead to an inconsistent system, which would be evidence that $\mathbf{y} \notin \langle Q \rangle$.

3. Find a set S so that S is linearly independent and $\mathcal{N}(A) = \langle S \rangle$, where $\mathcal{N}(A)$ is the null space of the matrix A below. (15 points)

$$A = \begin{bmatrix} 2 & 3 & 3 & 1 & 4 \\ 1 & 1 & -1 & -1 & -3 \\ 3 & 2 & -8 & -1 & 1 \end{bmatrix}$$

Solution: A direct application of Theorem BNS will provide the desired set. We require the reduced row-echelon form of A.

$$\begin{bmatrix} 2 & 3 & 3 & 1 & 4 \\ 1 & 1 & -1 & -1 & -3 \\ 3 & 2 & -8 & -1 & 1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & -6 & 0 & 3 \\ 0 & \boxed{1} & 5 & 0 & -2 \\ 0 & 0 & 0 & \boxed{1} & 4 \end{bmatrix}$$

The non-pivot columns have indicies $F = \{3,\, 5\}$. We build the desired set in two steps, first placing the requisite zeros and ones in locations based on F, then placing the negatives of the entries of columns 3 and 5 in the proper locations. This is all specified in Theorem BNS.

$$S = \left\{ \begin{bmatrix} \\ \\ 1 \\ \\ 0 \end{bmatrix},\ \begin{bmatrix} \\ \\ 0 \\ \\ 1 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} 6 \\ -5 \\ 1 \\ 0 \\ 0 \end{bmatrix},\ \begin{bmatrix} -3 \\ 2 \\ 0 \\ -4 \\ 1 \end{bmatrix} \right\}$$

4. Given the set T of column vectors below, find a set S so that S is linearly independent and $\langle S \rangle = \langle T \rangle$. (15 points)

$$T = \left\{ \begin{bmatrix} 1 \\ -1 \\ 3 \\ -1 \end{bmatrix},\ \begin{bmatrix} 2 \\ 1 \\ -1 \\ 2 \end{bmatrix},\ \begin{bmatrix} -4 \\ 1 \\ -5 \\ 0 \end{bmatrix},\ \begin{bmatrix} 7 \\ -1 \\ 7 \\ 1 \end{bmatrix} \right\}$$

Solution: Theorem BS will provide the desired answer, and requires examination of the reduced row-echelon form of a matrix whose columns are the vectors of T.

$$\begin{bmatrix} 1 & 2 & -4 & 7 \\ -1 & 1 & 1 & -1 \\ 3 & -1 & -5 & 7 \\ -1 & 2 & 0 & 1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & -2 & 3 \\ 0 & \boxed{1} & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since pivot columns reside in columns with indices $D = \{1,\, 2\}$, Theorem BS says we can form S with the first two vectors of T, so

$$T = \left\{ \begin{bmatrix} 1 \\ -1 \\ 3 \\ -1 \end{bmatrix},\ \begin{bmatrix} 2 \\ 1 \\ -1 \\ 2 \end{bmatrix} \right\}$$

5. Suppose that $\alpha \in \mathbb{C}$, $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$. Prove that $\alpha(\mathbf{x}+\mathbf{y}) = \alpha\mathbf{x} + \alpha\mathbf{y}$. (Note: this is Property DVAC of Theorem VSPCV, so you are being asked to do more than just quote this result from the book.) (15 points)

   Solution: This result says that two vectors are equal, so we must appeal to Definition CVE and look at each entry of these vectors in general,

$$\begin{aligned}
[\alpha(\mathbf{x}+\mathbf{y})]_i &= \alpha[\mathbf{x}+\mathbf{y}]_i && \text{Definition CVSM}\\
&= \alpha([\mathbf{x}]_i + [\mathbf{y}]_i) && \text{Definition CVA}\\
&= \alpha[\mathbf{x}]_i + \alpha[\mathbf{y}]_i && \text{Property DCN}\\
&= [\alpha\mathbf{x}]_i + [\alpha\mathbf{y}]_i && \text{Definition CVSM}\\
&= [\alpha\mathbf{x} + \alpha\mathbf{y}]_i && \text{Definition CVA}
\end{aligned}$$

   Since the two vectors agree in every entry, by Definition CVE, we say $\alpha(\mathbf{x}+\mathbf{y}) = \alpha\mathbf{x} + \alpha\mathbf{y}$.

6. Suppose that $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w} \in \mathbb{C}^n$, α, $\beta \in \mathbb{C}$ and $\mathbf{u}$ is orthogonal to both $\mathbf{v}$ and $\mathbf{w}$. Prove that $\mathbf{u}$ is orthogonal to $\alpha\mathbf{v} + \beta\mathbf{w}$. (15 points)

   Solution: Vectors are orthogonal if their inner product is zero, so we compute,

$$\begin{aligned}
\langle \alpha\mathbf{v} + \beta\mathbf{w},\, \mathbf{u} \rangle &= \langle \alpha\mathbf{v},\, \mathbf{u} \rangle + \langle \beta\mathbf{w},\, \mathbf{u} \rangle && \text{Theorem IPVA}\\
&= \alpha \langle \mathbf{v},\, \mathbf{u} \rangle + \beta \langle \mathbf{w},\, \mathbf{u} \rangle && \text{Theorem IPSM}\\
&= \alpha(0) + \beta(0) && \text{Definition OV}\\
&= 0
\end{aligned}$$

   So by Definition OV, $\mathbf{u}$ and $\alpha\mathbf{v} + \beta\mathbf{w}$ are an orthogonal pair of vectors.