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Quiz M Spring 2008

Show *all* of your work and *explain* your answers fully. There is a total of 100 possible points. If you use a calculator or software package on a problem be sure to write down both the input and output.

1. For the system below, compute the inverse of the nonsingular coefficient matrix by row-reducing the appropriate 2×4 matrix. Then use this inverse to compute the solution set of the system. (10 points)

$$\begin{aligned} 3x_1 + 2x_2 &= 5 \\ 11x_1 + 7x_2 &= 19 \end{aligned}$$

Solution: Augment the coefficient matrix of the system with the 2×2 identity matrix. By the combination of Theorem CINM and Theorem OSIS, we can find the inverse from the row-reduced version of this matrix,

$$\begin{bmatrix} 3 & 2 & 1 & 0 \\ 11 & 7 & 0 & 1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & -7 & 2 \\ 0 & \boxed{1} & 11 & -3 \end{bmatrix}$$

If we use Theorem SLEMM to rewrite the system as $A\mathbf{x} = \mathbf{b}$, then Theorem SNCM tells us the solution set contains a single vector,

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} -7 & 2 \\ 11 & -3 \end{bmatrix} \begin{bmatrix} 5 \\ 19 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

2. For the matrix A below, demonstrate the use of Theorem FS to compute the four indicated sets. (30 points)

$$A = \begin{bmatrix} -5 & 13 \\ 2 & -5 \\ -1 & 3 \end{bmatrix}$$

(a) Null space of A, $\mathcal{N}(A)$.

Solution: For all four parts of this problem, we need the extended echelon form of A (Definition EEF), and we extract the submatrices C and L.

$$\begin{bmatrix} -5 & 13 & 1 & 0 & 0 \\ 2 & -5 & 0 & 1 & 0 \\ -1 & 3 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 0 & 3 & 5 \\ 0 & \boxed{1} & 0 & 1 & 2 \\ 0 & 0 & \boxed{1} & 2 & -1 \end{bmatrix} \qquad C = \begin{bmatrix} \boxed{1} & 0 \\ 0 & \boxed{1} \end{bmatrix} \qquad L = \begin{bmatrix} \boxed{1} & 2 & -1 \end{bmatrix}$$

Using Theorem FS and Theorem BNS, $\mathcal{N}(A) = \mathcal{N}(C) = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$.

(b) Row space of A, $\mathcal{R}(A)$.

Solution: Using Theorem FS and Theorem BRS, $\mathcal{R}(A) = \mathcal{R}(C) = \left\langle \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \right\rangle = \mathbb{C}^2$.

(c) Column space of A, $\mathcal{C}(A)$.

Solution: Using Theorem FS and Theorem BNS, $\mathcal{C}(A) = \mathcal{N}(L) = \left\langle \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\} \right\rangle$.

(d) Left null space of A, $\mathcal{L}(A)$.

Solution: Using Theorem FS and Theorem BRS, $\mathcal{L}(A) = \mathcal{R}(L) = \left\langle \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \right\} \right\rangle$.

For Problems 3 and 4 use the following vectors and matrix. 15 points for each problem.

$$\mathbf{w} = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix} \qquad \mathbf{y} = \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \\ 2 \end{bmatrix} \qquad A = \begin{bmatrix} 2 & -1 & 1 & 0 & -3 \\ -3 & 2 & 0 & -1 & 4 \\ 4 & -3 & -1 & 2 & -5 \\ -5 & 4 & 2 & -3 & 6 \end{bmatrix}$$

3. (a) Prove directly (without using other parts of this problem) that $\mathbf{w}$ is an element of the column space of A, $\mathbf{w} \in \mathcal{C}(A)$.

Solution: By Theorem CSCS, $\mathbf{w} \in \mathcal{C}(A)$ if and only if $\mathcal{LS}(A, \mathbf{w})$ is consistent. We row-reduce the augmented matrix of this system

$$[A \,|\, \mathbf{w}] \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 2 & -1 & -2 & 3 \\ 0 & \boxed{1} & 3 & -2 & -1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

By Theorem RCLS this system is consistent, so $\mathbf{w} \in \mathcal{C}(A)$.

(b) Find a linearly independent set S such that the span of S is the column space of A, $\mathcal{C}(A) = \langle S \rangle$.

Solution: We can see the reduced row-echelon form of A in the first 5 columns of the row-reduced matrix in the previous part of this problem. In particular, $D = \{1, 2\}$. By Theorem BCS we can take columns 1 and 2 of A as column vectors and this set will fulfill the requirements of the question.

$$S = \left\{ \begin{bmatrix} 2 \\ -3 \\ 4 \\ -5 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \right\}$$

(c) Write $\mathbf{w}$ as a linear combination of the elements of S.

Solution: We can use the row-reduced matrix in part (a) to construct solutions to the linear system $\mathcal{LS}(A, \mathbf{w})$. By Theorem SLSLC these solutions will give rise to linear combinations of the columns of A that equal $\mathbf{x}$. In this case we want a linear combination of just the first two columns of A, so we want the scalars for the last three columns to be zero, i.e. $x_3 = x_4 = x_5 = 0$. Not coincidentally, the last three variables in the system are free, so we can just choose them to be zero. The result is that the first two variables are $x_1 = 3$ and $x_2 = 4$, so we have

$$3 \begin{bmatrix} 2 \\ -3 \\ 4 \\ -5 \end{bmatrix} + 4 \begin{bmatrix} -1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

4. (a) Prove directly (without using other parts of this problem) that $\mathbf{y}$ is an element of the row space of A, $\mathbf{y} \in \mathcal{R}(A)$.

Solution: By Definition RSM, $\mathbf{y} \in \mathcal{R}(A)$ if and only if $\mathbf{y} \in \mathcal{C}\left(A^t\right)$. We can test this by examining the consistency of $\mathcal{LS}\left(A^t,\,\mathbf{y}\right)$ (Theorem CSCS). We row-reduce the augmented matrix,

$$\left[A^t \mid \mathbf{y}\right] \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & -1 & 2 & -2 \\ 0 & \boxed{1} & -2 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

By Theorem RCLS this system is consistent, so $\mathbf{y} \in \mathcal{R}(A)$.

(b) Find a linearly independent set T such that the span of T is the row space of A, $\mathcal{R}(A) = \langle T \rangle$.

Solution: Theorem BRS tells us we can row-reduce A, and take the nonzero rows as column vectors to form T with the requested properties.

$$A \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 2 & -1 & -2 \\ 0 & \boxed{1} & 3 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad T = \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \\ -2 \end{bmatrix},\, \begin{bmatrix} 0 \\ 1 \\ 3 \\ -2 \\ -1 \end{bmatrix} \right\}$$

(c) Write $\mathbf{y}$ as a linear combination of the elements of T.

Solution: The first two entries of the each vector in T will suggest the scalars $a_1 = -1$, $a_1 = 0$,

$$\begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \\ 2 \end{bmatrix} = (-1) \begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \\ -2 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 1 \\ 3 \\ -2 \\ -1 \end{bmatrix}$$

5. Suppose that A is an $m \times n$ matrix and $\alpha \in \mathbb{C}$ is a scalar. Prove that $(\alpha A)^t = \alpha A^t$ (Note: this is Theorem TMSM, so you are being asked to do more than just quote this result from the book.) (15 points)

Solution: See the proof of Theorem TMSM in the book.

6. Prove the converse of Theorem NPNT: If A and B are nonsingular matrices of size n, then AB is a nonsingular matrix of size n. (15 points)

Solution: Suppose that $\mathbf{x} \in \mathbb{C}^n$ is a solution to $\mathcal{LS}(AB,\,\mathbf{0})$. Then

$$\begin{aligned} \mathbf{0} &= (AB)\,\mathbf{x} && \text{Theorem SLEMM} \\ &= A\,(B\mathbf{x}) && \text{Theorem MMA} \end{aligned}$$

By Theorem SLEMM, $B\mathbf{x}$ is a solution to $\mathcal{LS}(A,\,\mathbf{0})$, and by the definition of a nonsingular matrix (Definition NM), we conclude that $B\mathbf{x} = \mathbf{0}$. Now, by an entirely similar argument, the nonsingularity of B forces us to conclude that $\mathbf{x} = \mathbf{0}$. So the only solution to $\mathcal{LS}(AB,\,\mathbf{0})$ is the zero vector and we conclude that AB is nonsingular.