Math 290 Name: Key Dr. Beezer
Quiz VS Spring 2008

Show *all* of your work and *explain* your answers fully. There is a total of 90 possible points. If you use a calculator or software package on a problem be sure to write down both the input and output.

1. Prove that the set $C = \left\{2 + x + 3x^2,\ 1 - 2x - x^2,\ -3 + x + 2x^2\right\}$ is a basis of P_2, the vector space of polynomials with degree at most 2. (20 points)

 Solution: With an application of Theorem G, we can save a bit of work. Lets first establish that C is a linearly independent set, by starting with a relation of linear dependence (Definition RLD),

$$\begin{aligned}\mathbf{0} &= 0 + 0x + 0x^2\\ &= a_1\left(2 + x + 3x^2\right) + a_2\left(1 - 2x - x^2\right) + a_3\left(-3 + x + 2x^2\right)\\ &= (2a_1 + a_2 - 3a_3) + (a_1 - 2a_2 + a_3)\,x + (3a_1 - a_2 + 2a_3)\,x^2\end{aligned}$$

 The definition of equality in P_2 allows us to equate coefficients, which leads to a homogeneous system of three equations in the unknowns a_1, a_2, a_3, with coefficient matrix

$$A = \begin{bmatrix} 2 & 1 & -3\\ 1 & -2 & 1\\ 3 & -1 & 2\end{bmatrix}$$

 Check that A row-reduces to the identity matrix I_3 and so by Theorem NMRRI is a nonsingular matrix. Then by Definition NM, the only solution to the homogeneous system is $a_1 = a_2 = a_3 = 0$. By Definition LI, C is linearly independent.

 We know by Theorem DP that P_2 has dimension 3. Because C is a set of size 3, Theorem G tells us that C is a basis of P_2, sparing us the necessity of checking that C spans P_2.

2. Prove that D is a spanning set for the subspace X of M_{22}. (20 points)

$$D = \left\{\begin{bmatrix}1 & 0\\ 0 & 6\end{bmatrix}, \begin{bmatrix}0 & 1\\ 0 & -2\end{bmatrix}, \begin{bmatrix}0 & 0\\ 1 & 3\end{bmatrix}\right\} \qquad X = \left\{\begin{bmatrix}a & b\\ c & d\end{bmatrix}\,\middle|\, 6a - 2b + 3c - d = 0\right\} \subseteq M_{22}$$

 Solution: According to Definition TSVS we need to establish the set equality $\langle D\rangle = X$ (Definition SE). Check that each of three elements of D is an element of X (each passes the membership criteria), so by Definition SS, we have the first set inclusion, $\langle D\rangle \subseteq X$.

 To establish the second set inclusion, $X \subseteq \langle D\rangle$, we grab a generic element of X and ask if it can be written as a linear combination of the three elements of D? With the assumption that $6a - 2b + 3c - d = 0$ are there scalars α_1, α_2, α_3 such that

$$\begin{bmatrix}a & b\\ c & d\end{bmatrix} = \alpha_1\begin{bmatrix}1 & 0\\ 0 & 6\end{bmatrix} + \alpha_2\begin{bmatrix}0 & 1\\ 0 & -2\end{bmatrix} + \alpha_3\begin{bmatrix}0 & 0\\ 1 & 3\end{bmatrix}$$

 We massage the right-hand side,

$$= \begin{bmatrix}\alpha_1 & \alpha_2\\ \alpha_3 & 6\alpha_1 - 2\alpha_2 + 3\alpha_3\end{bmatrix}$$

 If we apply the definition of matrix equality (Definition ME) we get a system of four equations in the three variables α_1, α_2, α_3 which is represented by the augmented matrix,

$$\begin{bmatrix}1 & 0 & 0 & a\\ 0 & 1 & 0 & b\\ 0 & 0 & 1 & c\\ 6 & -2 & 3 & d\end{bmatrix}$$

This matrix is very nearly in reduced row-echelon form, we just need to perform (by hand) the operations $-6R_1 + R_4$, $2R_2 + R_4$, $-3R_3 + R_4$ and apply the knowledge that $6a - 2b + 3c - d = 0$,

$$\begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \\ 6 & -2 & 3 & d \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 0 & a \\ 0 & \boxed{1} & 0 & b \\ 0 & 0 & \boxed{1} & c \\ 0 & 0 & 0 & -6a+2b-3c+d \end{bmatrix} = \begin{bmatrix} \boxed{1} & 0 & 0 & a \\ 0 & \boxed{1} & 0 & b \\ 0 & 0 & \boxed{1} & c \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

By Theorem RCLS we see that the system is consistent, so there are scalars α_1, α_2, α_3, and so $X \subseteq \langle D \rangle$. By Definition SE, we have $X = \langle D \rangle$ and so D spans X.

Another approach to this problem is to begin with the set X and vary its description until a generic element is "obviously" written as a linear combination of the three elements in D. Note too that Theorem G does not apply since we have no advance knowledge of the dimension of X. (If we establish the linear independence of D then we could see that $\dim(X) = 3$.)

3. The set $Y = \left\{ a + bx + cx^2 + dx^3 \,\middle|\, a + b + 3c - 5d = 0,\ 2a - b + 3c + 4d = 0 \right\}$ is a subspace of P_3, the vector space of polynomials with degree at most 3 (you may assume this much). With complete justification, determine the dimension of Y. (20 points)

   Solution: We have to construct a basis of Y before we can (easily) compute the dimension. By manipulating the description of Y we can construct a spanning set, and then check the set for linear independence. First, consider the homogeneous system whose coefficient matrix we now row-reduce,

$$\begin{bmatrix} 1 & 1 & 3 & -5 \\ 2 & -1 & 3 & -4 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 2 & -3 \\ 0 & \boxed{1} & 1 & -2 \end{bmatrix}$$

   We can begin modifying Y by replacing the membership criteria with an equivalent homogeneous system, based on these two row-equivalent coefficient matrices,

$$\begin{aligned} Y &= \left\{ a + bx + cx^2 + dx^3 \,\middle|\, a + b + 3c - 5d = 0,\ 2a - b + 3c + 4d = 0 \right\} \\ &= \left\{ a + bx + cx^2 + dx^3 \,\middle|\, a + 2c - 3d = 0,\ b + c - 2d = 0 \right\} \\ &= \left\{ a + bx + cx^2 + dx^3 \,\middle|\, a = -2c + 3d,\ b = -c + 2d \right\} \\ &= \left\{ (-2c + 3d) + (-c + 2d)\,x + cx^2 + dx^3 \,\middle|\, c,\ d \in \mathbb{C} \right\} \\ &= \left\{ \left(-2c - cx + cx^2\right) + \left(3d + 2dx + dx^3\right) \,\middle|\, c,\ d \in \mathbb{C} \right\} \\ &= \left\{ c\left(-2 - x + x^2\right) + d\left(3 + 2x + x^3\right) \,\middle|\, c,\ d \in \mathbb{C} \right\} \\ &= \left\langle \left\{ -2 - x + x^2,\ 3 + 2x + x^3 \right\} \right\rangle \end{aligned}$$

   So $B = \left\{ -2 - x + x^2,\ 3 + 2x + x^3 \right\}$ is a spanning set for Y. Is B linearly independent? Let α_1 and α_2 be the unknown scalars in a relation of linear dependence,

$$\begin{aligned} \mathbf{0} &= 0 + 0x + 0x^2 + 0x^4 \\ &= \alpha_1\left(-2 - x + x^2\right) + \alpha_2\left(3 + 2x + x^3\right) \\ &= (-2\alpha_1 + 3\alpha_2) + (-\alpha_1 + 2\alpha_2) + \alpha_1 x^2 + \alpha_2 x^3 \end{aligned}$$

   The equality of the coefficients x^2 and x^3 implies that $\alpha_1 = \alpha_2 = 0$. So by Definition LI, the set B is linearly independent. By Definition B, we know B is a basis of Y with size 2, and so $\dim(Y) = 2$.

4. Illustrate the use of the three tests of Theorem TSS to prove that W is a subspace of $\mathbb{C}^3$. (15 points)

$$W = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \;\middle|\; 2x_1 + 5x_2 - 7x_3 = 0 \right\}$$

Solution: See your Section S class notes, Example SC3 and the solution to Exercise S.M20. In particular, the **existence** of the zero vector is not in question, but its **membership** in W could be checked as part of testing that W is non-empty. Also, be certain that your checks of the two types of closure are clearly written as establishing implications (clearly stated hypotheses, and their application in establishing the right conclusion).

5. Suppose that $B = \{\mathbf{x}_1,\, \mathbf{x}_2,\, \mathbf{x}_3,\, \ldots,\, \mathbf{x}_n\}$ is a basis for $\mathbb{C}^n$. Let A be a nonsingular matrix of size n and define $C = \{A\mathbf{x}_1,\, A\mathbf{x}_2,\, \ldots,\, A\mathbf{x}_n\}$.

Choose one of the following two statements (by circling it) and provide a proof. You **must** indicate which statement you have chosen to prove, or there will be no credit. (15 points)

(a) C is a linearly independent set.

(b) C is a spanning set of $\mathbb{C}^n$.

Solution: We provide proofs of both statements. As a practical matter, knowing that the dimension of $\mathbb{C}^n$ is n (Theorem DCM), we can apply Theorem G to prove one statement after we first prove the other. However this requires that we are certain that the set C has n distinct elements, which requires a short proof that turns on A being nonsingular.

(a) C is linearly independent. Work on a relation of linear dependence on C,

$$\begin{aligned}
\mathbf{0} &= a_1A\mathbf{x}_1 + a_2A\mathbf{x}_2 + a_3A\mathbf{x}_3 + \cdots + a_nA\mathbf{x}_n && \text{Definition RLD}\\
&= Aa_1\mathbf{x}_1 + Aa_2\mathbf{x}_2 + Aa_3\mathbf{x}_3 + \cdots + Aa_n\mathbf{x}_n && \text{Theorem MMSMM}\\
&= A\left(a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + a_3\mathbf{x}_3 + \cdots + a_n\mathbf{x}_n\right) && \text{Theorem MMDAA}
\end{aligned}$$

Since A is nonsingular, Definition NM and Theorem SLEMM allows us to conclude that

$$a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \cdots + a_n\mathbf{x}_n = \mathbf{0}$$

But this is a relation of linear dependence of the linearly independent set B, so the scalars are trivial, $a_1 = a_2 = a_3 = \cdots = a_n = 0$. By Definition LI, the set C is linearly independent.

(b) C spans $\mathbb{C}^n$. Given an arbitrary vector $\mathbf{y} \in \mathbb{C}^n$, can it be expressed as a linear combination of the vectors in C? Since A is a nonsingular matrix we can define the vector $\mathbf{w}$ to be the unique solution of the system $\mathcal{LS}(A,\, \mathbf{y})$ (Theorem NMUS). Since $\mathbf{w} \in \mathbb{C}^n$ we can write $\mathbf{w}$ as a linear combination of the vectors in the basis B. So there are scalars, $b_1,\, b_2,\, b_3,\, \ldots,\, b_n$ such that

$$\mathbf{w} = b_1\mathbf{x}_1 + b_2\mathbf{x}_2 + b_3\mathbf{x}_3 + \cdots + b_n\mathbf{x}_n$$

Then,

$$\begin{aligned}
\mathbf{y} &= A\mathbf{w} && \text{Theorem SLEMM}\\
&= A\left(b_1\mathbf{x}_1 + b_2\mathbf{x}_2 + b_3\mathbf{x}_3 + \cdots + b_n\mathbf{x}_n\right) && \text{Definition TSVS}\\
&= Ab_1\mathbf{x}_1 + Ab_2\mathbf{x}_2 + Ab_3\mathbf{x}_3 + \cdots + Ab_n\mathbf{x}_n && \text{Theorem MMDAA}\\
&= b_1A\mathbf{x}_1 + b_2A\mathbf{x}_2 + b_3A\mathbf{x}_3 + \cdots + b_nA\mathbf{x}_n && \text{Theorem MMSMM}
\end{aligned}$$

So we can write an arbitrary vector of $\mathbb{C}^n$ as a linear combination of the elements of C. In other words, C spans $\mathbb{C}^n$ (Definition TSVS).