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Quiz D&E Spring 2008

Show *all* of your work and *explain* your answers fully. There is a total of 90 possible points. If you use a calculator or software package on a problem be sure to write down both the input and output.

1. Given the matrix A below, find three matrices, E_1, E_2 and B such that (1) E_1, E_2 are elementary matrices, (2) B is row-equivalent to A, (3) B is in reduced row-echelon form, and (4) $B = E_2E_1A$. (15 points)

$$A = \begin{bmatrix} 2 & 4 & 2 \\ 3 & 6 & 3 \end{bmatrix}$$

Solution: E_1 and E_2 will be the elementary matrices that perform the necessary row operations (in the sense of Theorem EMDRO) to row-reduce A to B. So the bulk of this problem is to row-reduce A with just two row operations. Conveniently, our usual process for this (from the proof of Theorem REMEF) will yield the two row operations,

$$A = \begin{bmatrix} 2 & 4 & 2 \\ 3 & 6 & 3 \end{bmatrix} \xrightarrow{\frac{1}{2}R_1} \begin{bmatrix} 1 & 2 & 1 \\ 3 & 6 & 3 \end{bmatrix} \xrightarrow{-3R_1+R_2} \begin{bmatrix} \boxed{1} & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} = B$$

So E_1 will correspond to the first operation performed and E_2 will correspond to the second operation,

$$E_1 = E_1\left(\frac{1}{2}\right) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \qquad E_2 = E_{1,2}\left(-3\right) = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$$

2. Without using a calculator, find the eigenspaces of the matrix C below. (15 points)

$$C = \begin{bmatrix} -1 & 12 \\ -1 & 6 \end{bmatrix}$$

Solution: Before finding the eigenspaces of C, we need to know what the eigenvalues of C are. So we construct the characteristic polynomial of C,

$$\begin{aligned} p_C\left(x\right) &= \det\left(C - xI_2\right) && \text{Definition CP} \\ &= \begin{vmatrix} -1-x & 12 \\ -1 & 6-x \end{vmatrix} \\ &= (-1-x)(6-x) - (12)(-1) && \text{Definition DM} \\ &= -6 + x - 6x + x^2 + 12 = x^2 - 5x + 6 = (x-2)(x-3) \end{aligned}$$

By Theorem EMRCP, eigenvalues are roots of the characteristic polynomial, so the eigenvalues of C are $\lambda = 2$ and $\lambda = 3$. Now we can compute eigenspaces as null spaces via Theorem EMNS.

$\lambda = 2$

$$A - 2I_2 = \begin{bmatrix} -3 & 12 \\ -1 & 4 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & -4 \\ 0 & 0 \end{bmatrix}$$

$$\mathcal{E}_A\left(2\right) = \mathcal{N}(A - 2I_2) = \left\langle\left\{\begin{bmatrix} 4 \\ 1 \end{bmatrix}\right\}\right\rangle$$

$\lambda = 3$

$$A - 3I_2 = \begin{bmatrix} -4 & 12 \\ -1 & 3 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}$$

$$\mathcal{E}_A\left(3\right) = \mathcal{N}(A - 3I_2) = \left\langle\left\{\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right\}\right\rangle$$

3. Use the matrix A below for this problem. (30 points)

$$A = \begin{bmatrix} 8 & -6 & 6 & 6 \\ 24 & -16 & 12 & 18 \\ 9 & -6 & 5 & 6 \\ 6 & -3 & 0 & 5 \end{bmatrix}$$

(a) Use your calculator to find the eigenvalues of A (they should be integers), and state the algebraic multiplicity of each.

Solution: The two eigenvalues are $\lambda = -1$ and $\lambda = 2$. The algebraic multiplicities are $\alpha_A(-1) = 2$ and $\alpha_A(2) = 2$.

(b) Using your calculator only for row-reducing, find bases for the eigenspaces of A, and state the geometric multiplicity of each eigenvalue.

Solution:

$\lambda = -1$

$$A - (-1)I_4 = \begin{bmatrix} 9 & -6 & 6 & 6 \\ 24 & -15 & 12 & 18 \\ 9 & -6 & 6 & 6 \\ 6 & -3 & 0 & 6 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & -2 & 2 \\ 0 & \boxed{1} & -4 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathcal{E}_A(-1) = \mathcal{N}(A - (-1)I_4) = \left\langle \left\{ \begin{bmatrix} 2 \\ 4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\} \right\rangle$$

$\lambda = 2$

$$A - 2I_4 = \begin{bmatrix} 6 & -6 & 6 & 6 \\ 24 & -18 & 12 & 18 \\ 9 & -6 & 3 & 6 \\ 6 & -3 & 0 & 3 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & -1 & 0 \\ 0 & \boxed{1} & -2 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathcal{E}_A(2) = \mathcal{N}(A - 2I_4) = \left\langle \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\} \right\rangle$$

Each eigenspace is described as the span of a basis, and the geometric multiplicity of each eigenvalue is just the dimension of the eigenspace (Definition GME), so $\gamma_A(-1) = \gamma_A(2) = 2$.

(c) Explain clearly how your answers in parts (a) and (b) allow you to conclude that A is a diagonalizable matrix.

Solution: Since $\gamma_A(-1) = \alpha_A(-1)$ and $\gamma_A(2) = \alpha_A(2)$, Theorem DMFE tells us that A is diagonalizable.

(d) Find an invertible matrix S and a diagonal matrix D so that $A = S^{-1}DS$.

Solution: Theorem DC says there will be a linearly independent set of four eigenvectors of A. The proof of Theorem DC says we can use these eigenvectors as the columns of the desired matrix S and then D will have the associated eigenvalues along the diagonal. The proof of Theorem DMFE says we can get this set of four linearly independent eigenvectors by taking the union of the two bases for the two eigenspaces. So

$$S = \begin{bmatrix} 2 & -2 & 1 & 0 \\ 4 & -2 & 2 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

Note that it is not necessary to compute $S^{-1}AS$, we know by Theorem DC and Theorem DMFE that this computation will yield D. You could do the computation as a check, though.

4. We know that eigenspaces are subspaces; this is the substance of Theorem EMS. This question asks you to give a careful proof of the additive closure portion of the proof of that theorem. In other words, using only the definition of an eigenspace (Definition EM), carefully complete the second test of the three-part test for a subspace, Theorem TSS. (15 points)

   Solution: See the second part of the proof of Theorem EMS.

5. Let F be the $n \times n$ matrix where every entry is 1. More precisely, $[F]_{ij} = 1$ for $1 \leq i \leq n$, $1 \leq j \leq n$. Prove that $\lambda = n$ is an eigenvalue of F. (15 points)

   Solution: If you experiment with versions of F that are 2×2 or 3×3 you will perhaps discover that a column vector of all 1's will be an eigenvector of F for the eigenvalue n. So our approach is to simply demonstrate that $F\mathbf{x} = n\mathbf{x}$, where $\mathbf{x}$ is the all-1's vector.

   More carefully, define $\mathbf{x} \in \mathbb{C}^n$ by $[\mathbf{x}]_i = 1$ for $1 \leq i \leq n$. Then for $1 \leq i \leq n$,

$$
\begin{aligned}
[F\mathbf{x}]_i &= \sum_{k=1}^{n} [F]_{ik} [\mathbf{x}]_k && \text{Theorem EMP} \\
&= \sum_{k=1}^{n} (1)(1) && \text{Definition of } F,\ \mathbf{x} \\
&= \sum_{k=1}^{n} 1 = n = n(1) \\
&= n [\mathbf{x}]_i && \text{Definition of } \mathbf{x} \\
&= [n\mathbf{x}]_i && \text{Definition CVSM}
\end{aligned}
$$

   So by Definition CVE, we have $F\mathbf{x} = n\mathbf{x}$. Then by Definition EEM, $\lambda = n$ is an eigenvalue of F with eigenvector $\mathbf{x}$.