Math 290 Name: Key Dr. Beezer
Quiz LT Spring 2008

Show *all* of your work and *explain* your answers fully. There is a total of 90 possible points. If you use a calculator or software package on a problem be sure to write down both the input and output.

1. Consider the linear transformation T below, where P_2 is the vector space of polynomials with degree at most 2. (30 points)

$$T\colon P_2 \mapsto \mathbb{C}^4, \qquad T\left(a+bx+cx^2\right) = \begin{bmatrix} 2a+3b \\ a+b-c \\ 4a+5b-2c \\ a-b-5c \end{bmatrix}$$

(a) Find a basis for the kernel of T, $\mathcal{K}(T)$.

Solution: The kernel is defined as elements of the domain (polynomials) that T sends to the zero vector of the codomain (column vectors). So begin with this condition,

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = T\left(a+bx+cx^2\right) = \begin{bmatrix} 2a+3b \\ a+b-c \\ 4a+5b-2c \\ a-b-5c \end{bmatrix}$$

By the definition of column vector equality, Definition CVE, this can be expressed as a homogeneous system of four equations in three variables. With an eye to solving this system, we row-reduce the coefficient matrix,

$$\begin{bmatrix} 2 & 3 & 0 \\ 1 & 1 & -1 \\ 4 & 5 & -2 \\ 1 & -1 & -5 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & -3 \\ 0 & \boxed{1} & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

From this we can identify solutions for a, b and c to be of the form $a = 3c$, $b = -2c$ where c can take on any value (it is a free variable for the system). So kernel vectors are polynomials $3c+(-2c)x+cx^2 = c(3-2x+x^2)$ for all possible values of the scalar c. In particular, we can write,

$$\mathcal{K}(T) = \left\langle\left\{3-2x+x^2\right\}\right\rangle$$

(b) Find a basis for the range of T, $\mathcal{R}(T)$.

Solution: Theorem SSRLT gives a quick and easy way to build a spanning set of the range — just input the vectors of a spanning set for the domain and collect the outputs in a set. We can choose any spanning set of P_2 but a basis will be most efficient, and a nice basis will be even better. How about $B = \left\{1,\, x,\, x^2\right\}$? Then our spanning set for the range is

$$\left\{T(1),\, T(x),\, T\left(x^2\right)\right\} = \left\{ \begin{bmatrix} 2 \\ 1 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 5 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -2 \\ -5 \end{bmatrix} \right\}$$

There is no guarantee this set is a basis, as it may not be a linearly independent set. To replace the spanning set by a linearly independent spanning set, we can appeal to Theorem BCS since these vectors are column vectors. If we make these three vectors the columns of a matrix, we (not coincidentally) arrive at the same matrix in part (a). Since the matrix row-reduces to pivot columns with indices $D = \{1,\, 2\}$, we can just "keep" the first two columns as a basis. So we can write,

$$\mathcal{R}(T) = \left\langle \left\{ \begin{bmatrix} 2 \\ 1 \\ 4 \\ 1 \end{bmatrix},\, \begin{bmatrix} 3 \\ 1 \\ 5 \\ -1 \end{bmatrix} \right\} \right\rangle$$

(c) Compute the pre-image $T^{-1}\left(\begin{bmatrix} 5 \\ 2 \\ 3 \\ 0 \end{bmatrix}\right)$.

Solution: This question asks for those polynomials $a + bx + cx^2$ such that $T\left(a + bx + cx^2\right) = \begin{bmatrix} 5 \\ 2 \\ 3 \\ 0 \end{bmatrix}$. This condition is

$$\begin{bmatrix} 5 \\ 2 \\ 3 \\ 0 \end{bmatrix} = T\left(a + bx + cx^2\right) = \begin{bmatrix} 2a + 3b \\ a + b - c \\ 4a + 5b - 2c \\ a - b - 5c \end{bmatrix}$$

Equality of polynomials leads us to a system of four equations in three variables. To solve we row-reduce the augmented matrix,

$$\begin{bmatrix} 2 & 3 & 0 & 5 \\ 1 & 1 & -1 & 2 \\ 4 & 5 & -2 & 3 \\ 1 & -1 & -5 & 0 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & -3 & 0 \\ 0 & \boxed{1} & 2 & 0 \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

As an inconsistent system (Theorem RCLS) with no solutions, the preimage is empty.

(d) Compute the pre-image $T^{-1}\left(\begin{bmatrix} 1 \\ 1 \\ 3 \\ 3 \end{bmatrix}\right)$.

Solution: This question asks for those polynomials $a + bx + cx^2$ such that $T\left(a + bx + cx^2\right) = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 3 \end{bmatrix}$. This condition is

$$\begin{bmatrix} 1 \\ 1 \\ 3 \\ 3 \end{bmatrix} = T\left(a + bx + cx^2\right) = \begin{bmatrix} 2a + 3b \\ a + b - c \\ 4a + 5b - 2c \\ a - b - 5c \end{bmatrix}$$

Equality of polynomials leads us to a system of four equations in three variables. To solve we row-reduce the augmented matrix,

$$\begin{bmatrix} 2 & 3 & 0 & 1 \\ 1 & 1 & -1 & 1 \\ 4 & 5 & -2 & 3 \\ 1 & -1 & -5 & 3 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & -3 & 2 \\ 0 & \boxed{1} & 2 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

A single solution $a=2$, $b=-1$, $c=0$ implies that the polynomial $2-x$ is in the pre-image. We can combine this one element with the kernel from part (a) by Theorem KPI to obtain

$$T^{-1}\left(\begin{bmatrix}1\\1\\3\\3\end{bmatrix}\right) = (2-x) + \mathcal{K}(T) = (2-x) + \left\langle\left\{3-2x+x^2\right\}\right\rangle$$

2. Consider the linear transformation S below, where P_2 is the vector space of polynomials with degree at most 2. (30 points)

$$S\colon P_2 \mapsto P_2, \qquad S\left(a+bx+x^2\right) = (a+2b+c) + (2a+3b+3c)\,x + (a+b+c)\,x^2$$

(a) Establish that S is injective.

Solution: Examination of the kernel can help answer this question quickly. Membership of the polynomial $a+bx+cx^2$ in $\mathcal{K}(S)$ is equivalent to the condition,

$$0+0x+0x^2 = S\left(a+bx+cx^2\right) = (a+2b+c) + (2a+3b+3c)\,x + (a+b+c)\,x^2$$

This polynomial equality leads to a homogeneous system of three equations in three variables, where the coefficient matrix row-reduces as,

$$\begin{bmatrix}1&2&1\\2&3&3\\1&1&1\end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix}\boxed{1}&0&0\\0&\boxed{1}&0\\0&0&\boxed{1}\end{bmatrix}$$

The only solution is $a=b=c=0$, so the only polynomial in the kernel is the zero polynomial. Since the kernel is trivial, $\mathcal{K}(S)=\{\mathbf{0}\}$, by Theorem KILT, S is injective.

(b) Establish that S is surjective.

Solution: Since the kernel is trivial, the nullity is zero, $n\,(S)=0$. We can compute the rank of S,

$$\begin{aligned} r\,(S) &= \dim\left(P_2\right) - n\,(S) && \text{Theorem RPNDD}\\ &= 3-0=3 && \text{Theorem DP}\end{aligned}$$

By Theorem EDYES we can conclude the set equality $\mathcal{R}(S)=P_2$, and then by Theorem RSLT, we conclude that S is surjective.

(c) Explain how you can now easily determine that S is invertible.

Solution: Theorem ILTIS says that an injective and surjective linear transformation is invertible.

(d) Find a formula for S^{-1}.

Solution: Our strategy is to choose a basis for the codomain, find the pre-image of each of these elements and then find the linear transformation guaranteed by Theorem LTDB.

Our choice of basis will be $C=\left\{1,\,x,\,x^2\right\}$. As an example of the necessary computations, we will find the pre-image of x, $S^{-1}\,(x)$. $a+bx+cx^2 \in S^{-1}\,(x)$ is equivalent to the condition,

$$x = 0+1x+0x^2 = S\left(a+bx+cx^2\right) = (a+2b+c) + (2a+3b+3c)\,x + (a+b+c)\,x^2$$

This polynomial equality leads to a system of three equations in three variables, where the augmented matrix row-reduces as,

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 3 & 3 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 0 & -1 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 1 \end{bmatrix}$$

So the lone element of the pre-image is $-1 + x^2$, and we define the inverse linear transformation's value at x to be $S^{-1}(x) = -1 + x^2$.

For the other two basis elements, the computations are similar, except the vectors of constants in the reultant systems are a bit different for each. We would find,

$$S^{-1}(1) = x - x^2 \qquad\qquad S^{-1}\left(x^2\right) = 3 - x - x^2$$

Now, in the spirit of Theorem LTDB, we can determine a formula for the value of S^{-1} at any input,

$$\begin{aligned} S^{-1}\left(a + bx + cx^2\right) &= aS^{-1}(1) + bS^{-1}(x) + cS^{-1}\left(x^2\right) && \text{Theorem LTLC} \\ &= a\left(x - x^2\right) + b\left(-1 + x^2\right) + c\left(3 - x - x^2\right) \\ &= (-b + 3c) + (a - c)\,x + (-a + b - c)\,x^2 \end{aligned}$$

3. Suppose U is a vector space. A linear transformation $P\colon U \mapsto U$ is **idempotent** if $P \circ P = P$. Create and verify an example of a non-trivial idempotent linear transformation. Here, by trivial we mean any example that is the identity linear transformation on U, I_U, or a linear transformation that takes every input to the zero vector, $P(\mathbf{u}) = \mathbf{0}$ for all $\mathbf{u} \in U$. (15 points)

Solution: One possible example is, in a way, a mixture of two trivial examples. Consider

$$P\colon \mathbb{C}^2 \mapsto \mathbb{C}^2, \qquad P\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} a \\ 0 \end{bmatrix}$$

To verify, for all $\begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{C}^2$,

$$\begin{aligned} (P \circ P)\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) &= P\left(P\left(\begin{bmatrix} a \\ b \end{bmatrix}\right)\right) \\ &= P\left(\begin{bmatrix} a \\ 0 \end{bmatrix}\right) \\ &= \begin{bmatrix} a \\ 0 \end{bmatrix} \\ &= P\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) \end{aligned}$$

So, as functions, $P \circ P$ and P are equal.

4. Suppose that B is an $m \times n$ matrix and define the linear transformation $Q\colon \mathbb{C}^n \mapsto \mathbb{C}^m$ by $Q(\mathbf{x}) = B\mathbf{x}$. Give a careful proof that the null space of B is the kernel of Q, $\mathcal{N}(B) = \mathcal{K}(Q)$. (15 points)

Solution: See Solution ILT.T20