Math 290 Name: Key Dr. Beezer
Quiz R Spring 2008

Show *all* of your work and *explain* your answers fully. There is a total of 90 possible points. If you use a calculator or software package on a problem be sure to write down both the input and output.

1. Consider the linear transformation T below, where P_1 is the vector space of polynomials with degree at most 1. (35 points)

$$T\colon \mathbb{C}^2 \mapsto P_1, \qquad T\left(\begin{bmatrix}a\\b\end{bmatrix}\right) = (2a+b)+(a-b)\,x$$

(a) Compute the matrix representation of T, $M^T_{B,C}$, relative to the bases $B = \left\{\begin{bmatrix}1\\0\end{bmatrix},\,\begin{bmatrix}0\\1\end{bmatrix}\right\}$ and $C = \{1,\,x\}$.

Solution: Following Definition MR we compute

$$\rho_C\left(T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)\right) = \rho_C\left(2+x\right) = \begin{bmatrix}2\\1\end{bmatrix}$$
$$\rho_C\left(T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)\right) = \rho_C\left(1-x\right) = \begin{bmatrix}1\\-1\end{bmatrix}$$

So the representation is

$$M^T_{B,C} = \begin{bmatrix}2 & 1\\1 & -1\end{bmatrix}$$

(b) Compute the matrix representation of T, $M^T_{D,E}$, relative to the bases $D = \left\{\begin{bmatrix}2\\1\end{bmatrix},\,\begin{bmatrix}3\\2\end{bmatrix}\right\}$ and $E = \{1,\,1+x\}$.

Solution: Following Definition MR we compute

$$\rho_E\left(T\left(\begin{bmatrix}2\\1\end{bmatrix}\right)\right) = \rho_E\left(5+x\right) = \rho_E\left(4(1)+1(1+x)\right) = \begin{bmatrix}4\\1\end{bmatrix}$$
$$\rho_E\left(T\left(\begin{bmatrix}3\\2\end{bmatrix}\right)\right) = \rho_E\left(8+x\right) = \rho_E\left(7(1)+1(1+x)\right) = \begin{bmatrix}7\\1\end{bmatrix}$$

So the representation is

$$M^T_{D,E} = \begin{bmatrix}4 & 7\\1 & 1\end{bmatrix}$$

(c) Using the definition of T, compute $T\left(\begin{bmatrix}1\\2\end{bmatrix}\right)$.

Solution:

$$T\left(\begin{bmatrix}1\\2\end{bmatrix}\right) = (2(1)+2)+(1-2)\,x = 4-x$$

(d) Compute $T\left(\begin{bmatrix}1\\2\end{bmatrix}\right)$ with Theorem FTMR by computing $\rho_C^{-1}\left(M^T_{B,C}\rho_B\left(\begin{bmatrix}1\\2\end{bmatrix}\right)\right)$.

Solution:

$$\begin{aligned}
\rho_C^{-1}\left(M_{B,C}^T\rho_B\left(\begin{bmatrix}1\\2\end{bmatrix}\right)\right) &= \rho_C^{-1}\left(\begin{bmatrix}2 & 1\\1 & -1\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix}\right)\\
&= \rho_C^{-1}\left(\begin{bmatrix}4\\-1\end{bmatrix}\right)\\
&= 4(1)+(-1)x = 4-x
\end{aligned}$$

(e) Compute $T\left(\begin{bmatrix}1\\2\end{bmatrix}\right)$ with Theorem FTMR by computing $\rho_E^{-1}\left(M_{D,E}^T\rho_D\left(\begin{bmatrix}1\\2\end{bmatrix}\right)\right)$.

Solution:

$$\begin{aligned}
\rho_E^{-1}\left(M_{D,E}^T\rho_D\left(\begin{bmatrix}1\\2\end{bmatrix}\right)\right) &= \rho_E^{-1}\left(M_{D,E}^T\rho_D\left((-4)\begin{bmatrix}2\\1\end{bmatrix}+3\begin{bmatrix}3\\2\end{bmatrix}\right)\right)\\
&= \rho_E^{-1}\left(\begin{bmatrix}4 & 7\\1 & 1\end{bmatrix}\begin{bmatrix}-4\\3\end{bmatrix}\right)\\
&= \rho_E^{-1}\left(\begin{bmatrix}5\\-1\end{bmatrix}\right)\\
&= 5(1)+(-1)(1+x) = 4-x
\end{aligned}$$

2. Consider the linear transformation S below, where P_2 is the vector space of polynomials with degree at most 2. (30 points)

$$S\colon P_2 \mapsto P_2, \qquad S\left(a+bx+x^2\right) = (11a-24b+12c)+(6a-13b+6c)\,x+(2a-4b+c)\,x^2$$

(a) Build a matrix representation of S, using the same basis for both the domain and codomain.

Solution: With the freedom to pick any basis, we will keep it as simple as possible: $B\left\{1,\, x,\, x^2\right\}$. Following Definition MR.

$$\rho_B\left(S\left(1\right)\right) = \rho_B\left(11+6x+2x^2\right) = \begin{bmatrix}11\\6\\2\end{bmatrix}$$

$$\rho_B\left(S\left(x\right)\right) = \rho_B\left(-24-13x-4x^2\right) = \begin{bmatrix}-24\\-13\\-4\end{bmatrix}$$

$$\rho_B\left(S\left(x^2\right)\right) = \rho_B\left(12+6x+x^2\right) = \begin{bmatrix}12\\6\\1\end{bmatrix}$$

So the matrix representation is

$$M_{B,B}^S = \begin{bmatrix}11 & -24 & 12\\6 & -13 & 6\\2 & -4 & 1\end{bmatrix}$$

(b) Find a basis E for P_2 such that the matrix representation of S relative to E, $M_{E,E}^S$, is a diagonal representation.

Solution: The techniques illustrated in Subsection CB.CELT show that we can get eigenvalues and eigenvectors from any matrix representation. Eigenvalues are eigenvalues, but eigenvectors need to be interpreted ("un-coordinatized") relative to the choice of basis used in the representation. A basis of eigenvectors will provide the requested diagonal representation. So we use techniques from Chapter E to get the eigenvalues and eigenvectors of the matrix representation in part (a). For convenience, set $M = M^{S}_{B,B}$.

$$\lambda = 1 \qquad M - 1I_3 = \begin{bmatrix} 10 & -24 & 12 \\ 6 & -14 & 6 \\ 2 & -4 & 0 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & -6 \\ 0 & \boxed{1} & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathcal{E}_M(1) = \mathcal{N}(M - I_3) = \left\langle\left\{\begin{bmatrix} 6 \\ 3 \\ 1 \end{bmatrix}\right\}\right\rangle$$

$$\lambda = -1 \qquad M - (-1)I_3 = \begin{bmatrix} 12 & -24 & 12 \\ 6 & -12 & 6 \\ 2 & -4 & 2 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathcal{E}_M(-1) = \mathcal{N}(M - (-1)I_3) = \left\langle\left\{\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}\right\rangle$$

We can infer from the above that the algebraic and geometric multiplicity are equal for each eigenvalue, and from the proof of Theorem DMFE the three basis vectors above will together comprise a basis of $\mathbb{C}^3$. We just need to convert these column vectors into polynomials by un-coordinatizing relative to B,

$$\rho_B^{-1}\left(\begin{bmatrix} 6 \\ 3 \\ 1 \end{bmatrix}\right) = 6 + 3x + x^2 \qquad \rho_B^{-1}\left(\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}\right) = 2 + x \qquad \rho_B^{-1}\left(\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right) = -1 + x^2$$

So $E = \left\{6 + 3x + x^2,\, 2 + x,\, -1 + x^2\right\}$.

(c) S is invertible (you may assume this). Use the matrix representation in part (a) to construct a formula for S^{-1}.

Solution:

$$\begin{aligned}
S^{-1}\left(a + bx + cx^2\right) &= \rho_B^{-1}\left(M^{S^{-1}}_{B,B}\rho_B\left(a + bx + cx^2\right)\right) && \text{Theorem FTMR} \\
&= \rho_B^{-1}\left(\left(M^{S}_{B,B}\right)^{-1}\rho_B\left(a + bx + cx^2\right)\right) && \text{Theorem IMR} \\
&= \rho_B^{-1}\left(\begin{bmatrix} 11 & -24 & 12 \\ 6 & -13 & 6 \\ 2 & -4 & 1 \end{bmatrix}^{-1}\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) \\
&= \rho_B^{-1}\left(\begin{bmatrix} 11 & -24 & 12 \\ 6 & -13 & 6 \\ 2 & -4 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) \\
&= \rho_B^{-1}\left(\begin{bmatrix} 11a - 24b + 12c \\ 6a - 13b + 6c \\ 2a - 4b + c \end{bmatrix}\right) \\
&= (11a - 24b + 12c) + (6a - 13b + 6c)\,x + (2a - 4b + c)\,x^2
\end{aligned}$$

It may be a little startling to arrive at $S = S^{-1}$. Not 100% a coincidence — it's an artifact of keeping the computations simple.

3. Consider the linear transformation Q below. (25 points)

$$Q\colon M_{22} \mapsto P_2, \quad Q\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (5a + 5b + 10d) + (-2b - 2c - 2d)\,x + (a + 2b + c + 3d)\,x^2$$

(a) Find the matrix representation of Q relative to the bases B and C, $M^Q_{B,C}$.

$$B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \right\} \qquad C = \left\{1,\, 2 + x,\, 1 - x + x^2\right\}$$

Solution:

$$\rho_C\left(Q\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right)\right) = \rho_C\left(5 + 0x + x^2\right) = \rho_C\left(2\,(1) + 1\,(2 + x) + 1\,\left(1 - x + x^2\right)\right) = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

$$\rho_C\left(Q\left(\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}\right)\right) = \rho_C\left(0 - 2x + x^2\right) = \rho_C\left(1\,(1) + (-1)\,(2 + x) + 1\,\left(1 - x + x^2\right)\right) = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\rho_C\left(Q\left(\begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}\right)\right) = \rho_C\left(-10 + 2x + (-3)x^2\right) = \rho_C\left((-5)\,(1) + (-1)\,(2 + x) + (-3)\,\left(1 - x + x^2\right)\right) = \begin{bmatrix} -5 \\ -1 \\ -3 \end{bmatrix}$$

$$\rho_C\left(Q\left(\begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}\right)\right) = \rho_C\left(15 - 2x + 4x^2\right) = \rho_C\left(7\,(1) + (2)\,(2 + x) + 4\,\left(1 - x + x^2\right)\right) = \begin{bmatrix} 7 \\ 2 \\ 4 \end{bmatrix}$$

So the representation is

$$M^Q_{B,C} = \begin{bmatrix} 2 & 1 & -5 & 7 \\ 1 & -1 & -1 & 2 \\ 1 & 1 & -3 & 4 \end{bmatrix}$$

(b) Use the matrix representation from part (a) to compute a basis for the kernel of Q, $\mathcal{K}(Q)$.

Solution: By Theorem KNSI the kernel of Q is isomorphic to the null space of a matrix representation via the isomorphism that is vector representation relative to the basis chosen for the domain. We will turn this around and compute the null space of the matrix representation (via Theorem BNS) and then use the inverse of the isomorphism, which is "un-coordinatization."

Row-reduce the matrix representation from part (a),

$$M^Q_{B,C} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & -2 & 3 \\ 0 & \boxed{1} & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So, by Theorem BNS, a basis for the null space is

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

These two vectors are not elements of the domain, they are vector representations of matrices in M_{22} relative to the basis B, to get a basis of the kernel we will apply ρ_B^{-1}, to obtain the two basis vectors,

$$\rho_B^{-1}\left(\begin{bmatrix}2\\1\\1\\0\end{bmatrix}\right) = 2\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} + 1\begin{bmatrix}-1 & 1\\0 & 0\end{bmatrix} + 1\begin{bmatrix}0 & -2\\1 & 0\end{bmatrix} + 0\begin{bmatrix}0 & 1\\-1 & 1\end{bmatrix} = \begin{bmatrix}1 & -1\\1 & 0\end{bmatrix}$$

$$\rho_B^{-1}\left(\begin{bmatrix}-3\\-1\\0\\1\end{bmatrix}\right) = (-3)\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} + (-1)\begin{bmatrix}-1 & 1\\0 & 0\end{bmatrix} + 0\begin{bmatrix}0 & -2\\1 & 0\end{bmatrix} + 1\begin{bmatrix}0 & 1\\-1 & 1\end{bmatrix} = \begin{bmatrix}-2 & 0\\-1 & 1\end{bmatrix}$$

(c) Perform a partial check on your answer in (b) by evaluating Q with a vector from your basis.

Solution:

$$\begin{aligned} Q\left(\begin{bmatrix}-2 & 0\\-1 & 1\end{bmatrix}\right) &= (5(-2) + 5(0) + 10(1)) + (-2(0) - 2(-1) - 2(1))\,x + ((-2) + 2(0) + (-1) + 3(1))\,x^2 \\ &= 0 + 0x + 0x^2 \end{aligned}$$