Math 290 Name: Key Dr. Beezer
Quiz SLE Code: Spring 2010

Show *all* of your work and *explain* your answers fully. There is a total of 90 possible points. If you use a calculator or software package on a problem be sure to write down both the input and output.

1. Find all solutions to the system of equations below. Express the solutions as a set of column vectors. (15 points)

$$\begin{aligned} 2x_1 + 4x_2 + x_3 + 13x_4 &= -1 \\ 2x_1 + 4x_2 + 5x_3 + 25x_4 &= 0 \\ -2x_1 - 4x_2 \qquad - 10x_4 &= 1 \end{aligned}$$

Solution: We form the augmented matrix of the system and row-reduce

$$\begin{bmatrix} 2 & 4 & 1 & 13 & -1 \\ 2 & 4 & 5 & 25 & 0 \\ -2 & -4 & 0 & -10 & 1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 2 & 0 & 5 & 0 \\ 0 & 0 & \boxed{1} & 3 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \end{bmatrix}$$

With a leading 1 in the final column, this system is inconsistent (Theorem RCLS). Written as a set, $S = \{\,\}$.

2. Find all solutions to the system of equations below. Express the solutions as a set of column vectors. (15 points)

$$\begin{aligned} x_1 - 3x_2 - 15x_3 &= 7 \\ -x_1 \qquad + 6x_3 &= -4 \\ x_1 + 2x_2 \qquad &= 2 \end{aligned}$$

Solution: We form the augmented matrix of the system and row-reduce,

$$\begin{bmatrix} 1 & -3 & -15 & 7 \\ -1 & 0 & 6 & -4 \\ 1 & 2 & 0 & 2 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & -6 & 4 \\ 0 & \boxed{1} & 3 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

With no leading 1 in the final column, this system is consistent (Theorem RCLS). There are $n = 3$ variables in the system and $r = 2$ non-zero rows in the row-reduced matrix. By Theorem FVCS, there are $n-r = 3-2 = 1$ free variables and we therefore know the solution set is infinite. Forming the system of equations represented by the row-reduced matrix, we express each dependent variable (x_1, x_2) in terms of the one free variable (x_3). Writing the solution set as a set of column vectors,

$$S = \left\{ \left. \begin{bmatrix} 4 + 6x_3 \\ -1 - 3x_3 \\ x_3 \end{bmatrix} \right| x_3 \in \mathbb{C} \right\}$$

3. Find a matrix that is row-equivalent to C and in reduced row-echelon form. Perform the necessary computations by hand, not with a calculator, and show your intermediate steps along with the row operations you've performed. (15 points)

$$C = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 0 & -2 \\ 3 & 1 & 3 \end{bmatrix}$$

Solution: Following the algorithm of Theorem REMEF, and working to create pivot columns from left to right, we have

$$\begin{bmatrix} 1 & 1 & -1 \\ -1 & 0 & -2 \\ 3 & 1 & 3 \end{bmatrix} \xrightarrow{-1R_1+R_2} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -3 \\ 3 & 1 & 3 \end{bmatrix} \xrightarrow{-3R_1+R_3}$$

$$\begin{bmatrix} \boxed{1} & 1 & -1 \\ 0 & 1 & -3 \\ 0 & -2 & 6 \end{bmatrix} \xrightarrow{-1R_2+R_1} \begin{bmatrix} \boxed{1} & 0 & 2 \\ 0 & 1 & -3 \\ 0 & -2 & 6 \end{bmatrix} \xrightarrow{-2R_2+R_3} \begin{bmatrix} \boxed{1} & 0 & 2 \\ 0 & \boxed{1} & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

4. Find the null space of the matrix E below. (15 points)

$$E = \begin{bmatrix} 1 & 0 & 2 & -1 \\ -2 & 1 & -1 & 6 \\ 1 & -1 & -1 & -5 \\ -1 & -1 & -5 & -3 \end{bmatrix}$$

Solution: We now view the matrix as the coefficient matrix of the homogeneous system $\mathcal{LS}(E, \mathbf{0})$ and row-reduce the matrix,

$$\begin{bmatrix} 1 & 0 & 2 & -1 \\ -2 & 1 & -1 & 6 \\ 1 & -1 & -1 & -5 \\ -1 & -1 & -5 & -3 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 2 & -1 \\ 0 & \boxed{1} & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We knew ahead of time that this system would be consistent (Theorem HSC), but we can now see there are $n - r = 4 - 2 = 2$ free variables, namely x_3 and x_4 since $F = \{3, 4, 5\}$ (Theorem FVCS). Based on this analysis, we can rearrange the equations associated with each nonzero row of the reduced row-echelon form into an expression for the lone dependent variable as a function of the free variables. Recall that if we worked with an augmented matrix, we would have a fifth column full of zeros. We arrive at the solution set to the homogeneous system, which is the null space of the matrix by Definition NSM,

$$\mathcal{N}(B) = \left\{ \begin{bmatrix} -2x_3 + x_4 \\ -3x_3 - 4x_4 \\ x_3 \\ x_4 \end{bmatrix} \middle| \, x_3, \, x_4 \in \mathbb{C} \right\}$$

5. For each of the systems of linear equations described below, say as much as you can, with reasons, about the nature of the solution set. (15 points)

   (a) Consistent, 12 equations, 5 variables.

   Solution: Definition CS rules out an empty solution set. There is not enough information from the variable and equation counts to infer any more, so there is a unique solution, or infinitely many solutions by Theorem PSSLS.

   (b) Homogeneous, 8 equations, 17 variables.

   Solution: Theorem HMVEI tells us this homogeneous system has infinitely many solutions. One of these solutions will be the trivial solution (Definition TSHSE).

   (c) 10 equations and 10 variables.

   Solution: We can consider if the square coefficient matrix is singular or nonsingular. If this matrix is nonsingular, then Theorem NMUS tells us there is a unique solution. If the coefficient matrix is singular, there is no solution, or infinitely many. So the information on equations and variables provides no evidence about the nature of the solution set.

6. Suppose that A is a singular matrix. Prove that the homogeneous system $\mathcal{LS}(A, \mathbf{0})$ has infinitely many solutions. Provide a careful and fully-justified proof. (15 points)

   Solution: As a homogeneous system, the solution set cannot be empty by Theorem HSC. By Theorem PSSLS,the solution set has a single solution or infinitely many solutions. We show that a single solution is impossible with a proof by contradiction (see Technique CD).

   As a homogeneous system, a single solution must be the trivial solution (Definition TSHSE). However, this now says the matix is nonsingular (Definition NM), a contradiction. The only possibility left is that there are infinitely many solutions.