Math 290 Name: Key Dr. Beezer
Quiz V Spring 2010

Show *all* of your work and *explain* your answers fully. There is a total of 90 possible points. If you use a calculator or software package on a problem be sure to write down both the input and output.

1. Give the vector $\mathbf{y}$ and the set $S \subseteq \mathbb{C}^4$ below, determine if $\mathbf{y} \in S$. (15 points)

$$\mathbf{y} = \begin{bmatrix} 19 \\ 29 \\ 2 \\ 10 \end{bmatrix} \qquad S = \left\{ \begin{bmatrix} 2 \\ -2 \\ 7 \\ -7 \end{bmatrix}, \begin{bmatrix} 0 \\ -4 \\ 4 \\ -8 \end{bmatrix}, \begin{bmatrix} 5 \\ 7 \\ 0 \\ 0 \end{bmatrix} \right\}$$

Solution: According to Definition SSCV, the vector $\mathbf{y}$ will be a member of the span, $\langle S \rangle$, if there are scalars a_1, a_2, a_3 such that

$$\begin{bmatrix} 19 \\ 29 \\ 2 \\ 10 \end{bmatrix} = a_1 \begin{bmatrix} 2 \\ -2 \\ 7 \\ -7 \end{bmatrix} + a_2 \begin{bmatrix} 0 \\ -4 \\ 4 \\ -8 \end{bmatrix} + a_3 \begin{bmatrix} 5 \\ 7 \\ 0 \\ 0 \end{bmatrix}$$

By Theorem SLSLC a solution to this vector equation is a solution to the linear system with the augmented matrix below.

$$\begin{bmatrix} 2 & 0 & 5 & 19 \\ -2 & -4 & 7 & 29 \\ 7 & 4 & 0 & 2 \\ -7 & -8 & 0 & 10 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 0 & 2 \\ 0 & \boxed{1} & 0 & -3 \\ 0 & 0 & \boxed{1} & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

With no leading 1 in the final column of the row-eqivalent matrix in reduced row-echelon form, we know the system has a solution (Theorem RCLS), and thus $\mathbf{y}$ is a member of the span. Notice that to answer the yes/no question about membership, it is not necessary to actually find a solution (even if it is obvious here).

2. Determine if the set $Q \subseteq \mathbb{C}^4$ below is linearly independent. (15 points)

$$Q = \left\{ \begin{bmatrix} -4 \\ -3 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 6 \\ 8 \\ -3 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 0 \\ 4 \end{bmatrix} \right\}$$

Solution: The four vectors in Q, as columns of a matrix, will form a 4×4 matrix, A. Thus, by Theorem NME2, Q is linearly independent if and only if A is nonsingular. So we form A and find the row-equivalent matrix in reduced row-echelon form,

$$A = \begin{bmatrix} -4 & 6 & 3 & 1 \\ -3 & 8 & 4 & 5 \\ 0 & -3 & -2 & 0 \\ 3 & 6 & 2 & 4 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & \boxed{1} \end{bmatrix}$$

Since A row-reduces to the 4×4 identity matrix I_4, Theorem NMRRI tells us A is nonsingular and thus Q is linearly independent.

3. Express the null space of the matrix A below as the span of a linearly independent set of vectors. (15 points)

$$A = \begin{bmatrix} 3 & 4 & 3 & 6 & 5 \\ 3 & 3 & 2 & 5 & 5 \\ 4 & 0 & 3 & 20 & -2 \end{bmatrix}$$

Solution: Theorem BNS gives an explicit description of the vectors needed for such a set, once you have a row-equivalent matrix in reduced row-echelon form,

$$\begin{bmatrix} 3 & 4 & 3 & 6 & 5 \\ 3 & 3 & 2 & 5 & 5 \\ 4 & 0 & 3 & 20 & -2 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 0 & 2 & 1 \\ 0 & \boxed{1} & 0 & -3 & 2 \\ 0 & 0 & \boxed{1} & 4 & -2 \end{bmatrix}$$

With $n - r = 5 - 3$ non-pivot columns, we will construct two vectors. We demonstrate this construction in two steps.

$$S = \{\mathbf{z}_1,\, \mathbf{z}_2\} = \left\{ \begin{bmatrix} \\ \\ \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \\ \\ \\ 0 \\ 1 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} -2 \\ 3 \\ -4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Then an appeal to Theorem BNS gives us the result that S is linearly independent, and $\mathcal{N}(A) = \langle S \rangle$. Notice that none of the above makes any reference to a system of equations. You could describe the null space as the solution set to a homogeneous system, then write the solutions using vectors from Theorem VFSLS, but that would be more work than is necessary.

4. Given the set $T \subseteq \mathbb{C}^3$ below, find a set R that is a subset of T such that (a) R is linearly independent, (b) $\langle R \rangle = \langle T \rangle$. (15 points)

$$T = \left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ -9 \\ -6 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 9 \end{bmatrix} \right\}$$

Solution: Theorem BNS gives us all this with little effort. The theorem requires making the vectors the column of a matrix and finding the row-equivalent matrix in reduced row-echelon form.

$$\begin{bmatrix} 1 & -3 & 0 & 0 & 2 \\ 3 & -9 & 1 & 0 & 3 \\ 2 & -6 & 1 & 4 & 9 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & -3 & 0 & 0 & 2 \\ 0 & 0 & \boxed{1} & 0 & -3 \\ 0 & 0 & 0 & \boxed{1} & 2 \end{bmatrix}$$

We see the pivot columns as those with indices in $D = \{1, 3, 4\}$. Theorem BS says we can grab the first, third and fourth vectors from the set, to wit

$$R = \left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix} \right\}$$

and R will be linearly independent with $\langle R \rangle = \langle T \rangle$.

5. Give a careful proof of the following equality for vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w} \in \mathbb{C}^n$. (15 points)

$$\langle \mathbf{u}+\mathbf{v},\, \mathbf{w}\rangle = \langle \mathbf{u},\, \mathbf{w}\rangle + \langle \mathbf{v},\, \mathbf{w}\rangle$$

Solution:

$$\begin{aligned}
\langle \mathbf{u}+\mathbf{v},\, \mathbf{w}\rangle &= \sum_{i=1}^{n} [\mathbf{u}+\mathbf{v}]_i\, \overline{[\mathbf{w}]_i} && \text{Definition IP}\\
&= \sum_{i=1}^{n} ([\mathbf{u}]_i + [\mathbf{u}]_i)\, \overline{[\mathbf{w}]_i} && \text{Definition CVA}\\
&= \sum_{i=1}^{n} [\mathbf{u}]_i\, \overline{[\mathbf{w}]_i} + [\mathbf{u}]_i\, \overline{[\mathbf{w}]_i} && \text{Property DCN}\\
&= \sum_{i=1}^{n} [\mathbf{u}]_i\, \overline{[\mathbf{w}]_i} + \sum_{i=1}^{n} [\mathbf{u}]_i\, \overline{[\mathbf{w}]_i} && \text{Property CACN}\\
&= \langle \mathbf{u},\, \mathbf{w}\rangle + \langle \mathbf{v},\, \mathbf{w}\rangle && \text{Definition IP}
\end{aligned}$$

6. Suppose that $S = \{\mathbf{w},\, \mathbf{x},\, \mathbf{y}\} \subseteq \mathbb{C}^n$ is a linearly independent set of vectors. Determine if the set $T = \{2\mathbf{w}+3\mathbf{x},\, (-1)\mathbf{x}+2\mathbf{y},\, 2\mathbf{w}+2\mathbf{x}+2\mathbf{y}\}$ is linearly independent. (15 points)

Solution: Begin with a relation of linear dependence on T,

$$\mathbf{0} = a_1(2\mathbf{w}+3\mathbf{x}) + a_2((-1)\mathbf{x}+2\mathbf{y}) + a_3(2\mathbf{w}+2\mathbf{x}+2\mathbf{y})$$

and rearrange using properties in Theorem VSPCV,

$$\mathbf{0} = (2a_1+2a_3)\,\mathbf{w} + (3a_1+(-1)a_2+2a_3)\,\mathbf{x} + (2a_2+2a_3)\,\mathbf{y}$$

Because S is linearly independent, we are forced to conclude the following equations (Definition LICV), which together form a homogeneous ssytem of equations.

$$\begin{aligned}
2a_1 + 2a_3 &= 0\\
3a_1 + (-1)a_2 + 2a_3 &= 0\\
2a_2 + 2a_3 &= 0
\end{aligned}$$

A nontrivial solution to this system is $a_1 = 1$, $a_2 = 1$, $a_3 = -1$, which is enough to establish that T is linearly dependent (Definition LICV).