Math 290 Name: Dr. Beezer
Exam 4 Spring 2010
Chapter VS

Show *all* of your work and *explain* your answers fully. There is a total of 100 possible points.

1. For the matrix A below, compute the dimension of the column space, the row space, the null space and the left null space. (10 points)

$$A = \begin{bmatrix} 1 & 2 & 4 & 1 & 5 \\ -1 & 2 & 0 & 3 & 3 \\ 3 & -2 & 4 & -5 & -1 \end{bmatrix}$$

Solution: To determine these dimensions, we need only compute the rank of the matrix: r, the number of nonzero rows, the number of pivot column, the number of leading ones in the reduced row-echelon form.

$$\begin{bmatrix} 1 & 2 & 4 & 1 & 5 \\ -1 & 2 & 0 & 3 & 3 \\ 3 & -2 & 4 & -5 & -1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 2 & -1 & 1 \\ 0 & \boxed{1} & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus $r = 2$, and A is a matrix with $m = 3$ rows and $n = 5$ columns, so by Theorem DFS we have

$$\mathcal{N}(A) = n - r = 3 \qquad \mathcal{C}(A) = r = 2 \qquad \mathcal{R}(A) = r = 2 \qquad \mathcal{L}(A) = m - r = 1$$

2. In the vector space M_{22} of 2×2 matrices, determine if the set L below is linearly independent. (10 points)

$$L = \left\{ \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}, \begin{bmatrix} 4 & -2 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} -2 & -1 \\ 1 & -1 \end{bmatrix} \right\}$$

Solution: Begin with a relation of linear dependence (Definition RLD)

$$a_1 \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix} + a_2 \begin{bmatrix} 4 & -2 \\ 1 & 2 \end{bmatrix} + a_3 \begin{bmatrix} -2 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Vector space operations and equality in M_{22} lead to a homogeneous system of four equations in three unknowns, with a coefficient matrix that we now row-reduce

$$\begin{bmatrix} 2 & 4 & -2 \\ -1 & -2 & -1 \\ 3 & 1 & 1 \\ 1 & 2 & -1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & \frac{3}{5} \\ 0 & \boxed{1} & -\frac{4}{5} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

With a free variable, the consistent system has infinitely many solutions, and any non-trivial solution (like $a_1 = -3$, $a_2 = 4$, $a_3 = 5$) will provide a non-trivial relation of linear dependence. Thus L is linearly dependent (Definition LI).

3. In P_2, the vector space of 2×2 matrices, determine if the set K below spans P_2. (10 points)

$K = \left\{2x^2 - x + 3,\, x^2 + x + 1,\, 3x^2 + 2\right\}$

Solution: This could be established directly, but might be easier with an application of Theorem G. We know $\dim(P_2) = 3$ by Theorem DP. And K has size 3, so by Theorem G we can check linear independence and get the spanning property for "free." So, as in the previous question begin with a relation of linear dependence (Definition RLD)

$$a_1(2x^2 - x + 3) + a_2(x^2 + x + 1) + a_3(3x^2 + 2) = 0x^2 + 0x + 0$$

Vector space operations and equality in P_2 lead to a homogeneous system of three equations in three unknowns, with a coefficient matrix that we now row-reduce

$$\begin{bmatrix} 2 & 1 & 3 \\ -1 & 1 & 0 \\ 3 & 1 & 2 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \end{bmatrix}$$

With only a trivial solution to this system, we conclude that $a_1 = a_2 = a_3 = 0$. Thus L is linearly independent (Definition LI).

4. Consider the following subspace of P_2, the set of polynomials of degree 2.

$U = \left\{ ax^2 + bx + c \,\middle|\, a - 4b + c = 0 \right\}$

Solution:

(a) Construct a basis for U and include the necessary details to prove that your set really is a basis. (20 points)

Solution: The defining equation for U can be rewritten as $a = 4b - c$. This observation, and vector space operations in P_2 allows us to rewrite a generic elment of U as follows:

$$\begin{aligned} ax^2 + bx + c &= (4b - c)x^2 + bx + c \\ &= (4bx^2 + bx) + (-cx^2 + c) \\ &= b(4x^2 + x) + c(-x^2 + 1) \end{aligned}$$

Allowing b, c to take on any possible values says that $D = \left\{4x^2 + x,\, -x^2 + 1\right\}$ is a spanning set for P_2. To be a basis, we need to check the linear independence of the set D. See several other problems in this exam for the technique — it turns out that D is also linearly independent, hence a basis (Definition D).

(b) What is the dimension of U? (6 points)

Solution: With a basis of size 2, $\dim(U) = 2$.

(c) Prove that $C = \left\{x^2 + 2x + 7,\, 2x^2 + 3x + 10\right\}$ is another basis for U. (14 points)

Solution: Since U has dimension 2, the set C has the right size to be another basis. By Theorem G, if C is linearly independent, then it is a basis. We check this in the usual manner.

Begin with a relation of linear dependence (Definition RLD)

$$a_1(x^2 + 2x + 7) + a_2(2x^2 + 3x + 10) = 0x^2 + 0x + 0$$

Vector space operations and equality in P_2 lead to a homogeneous system of three equations in two unknowns, with a coefficient matrix that we now row-reduce

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 7 & 10 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 \\ 0 & \boxed{1} \\ 0 & 0 \end{bmatrix}$$

With only a trivial solution to this system, we conclude that $a_1 = a_2 = 0$. Thus C is linearly independent (Definition LI) and therefore also a basis of U.

5. Suppose V is a vector space and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$. Prove that if $\mathbf{u} + \mathbf{w} = \mathbf{v} + \mathbf{w}$ then $\mathbf{u} = \mathbf{v}$. Give precise reasons for each step of your proof. (15 points)

   Solution: See Solution VS.T21, which is nearly identical to the proof you could give here.

6. Suppose that $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for the vector space V. Prove that $C = \{\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3, \mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_3\}$ is also a basis for V. (15 points)

   Solution: That B is a basis implies $\dim(V) = 3$ (Definition D). C is a set of size 3, and so has the right size to be a basis for V. Check to see if C is linearly independent.

$$\begin{aligned}\mathbf{0} &= a(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3) + b(\mathbf{v}_1 + \mathbf{v}_2) + c\mathbf{v}_3 \\ &= a\mathbf{v_1} + (a+b)\mathbf{v}_2 + (a+b+c)\mathbf{v}_3\end{aligned}$$

   This is a relation of linear dependence on B (a linearly independent set) where the scalars are a, $a+b$, $a+b+c$, so they **must** each be zero. So we have

$$\begin{aligned}a &= 0 \\ a+b &= 0 \\ a+b+c &= 0\end{aligned}$$

   This homogeneous system has only a trvivial solution, so the original relation of linear dependence on C has only a trivial solution, and hence is linearly independent. So C is a basis for V.