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Exam 6 Spring 2010
Chapter LT

Show *all* of your work and *explain* your answers fully. There is a total of 100 possible points.

1. Consider the following questions for the linear transformation T defined below. (30 points)

$$T\colon \mathbb{C}^3 \to \mathbb{C}^2, \quad T\left(\begin{bmatrix} a\\ b\\ c\end{bmatrix}\right) = \begin{bmatrix} 2a+b-3c\\ -a+b+3c\end{bmatrix}$$

Solution:

(a) Find at least one element of the pre-image $T^{-1}\left(\begin{bmatrix} -1\\ 2\end{bmatrix}\right)$.

Solution: We desire an input vector $\begin{bmatrix} a\\ b\\ c\end{bmatrix}$ such that

$$T\left(\begin{bmatrix} a\\ b\\ c\end{bmatrix}\right) = \begin{bmatrix} 2a+b-3c\\ -a+b+3c\end{bmatrix} = \begin{bmatrix} -1\\ 2\end{bmatrix}$$

This leads a system of equations, with the augmented matrix below,

$$\begin{bmatrix} 2 & 1 & -3 & -1\\ -1 & 1 & 3 & 2\end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & -2 & -1\\ 0 & \boxed{1} & 1 & 1\end{bmatrix}$$

Setting the free variable to zero, we find a solution: $\begin{bmatrix} a\\ b\\ c\end{bmatrix} = \begin{bmatrix} -1\\ 1\\ 0\end{bmatrix}$ that is an element of the pre-image. There are, of course, other answers depending on your choice for the free variable.

(b) Is T injective? Justify your answer without using your answer to the next part of this problem.

Solution: The kernel of T is the pre-image of zero, so similar to the previous part of this problem, the condition that a domain element be in the kernel is equivalent to being a solution to the homogeneous system with augmented matrix,

$$\begin{bmatrix} 2 & 1 & -3 & 0\\ -1 & 1 & 3 & 0\end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & -2 & 0\\ 0 & \boxed{1} & 1 & 0\end{bmatrix}$$

With a free variable, we see the kernel is non-trivial and by Theorem KILT the linear transformation is not injective.

(c) Find two different elements of the domain, such that when used to evaluate T produce the same output. Or explain why this is not possible.

Solution: We could find two elements in the kernel (both produce the zero vector as an output), or we can find a second element of the pre-image in the first part of this question. Pursuing this second approach, set the free variable to 1 and obtain the second input $\begin{bmatrix} a\\ b\\ c\end{bmatrix} = \begin{bmatrix} 1\\ 0\\ 1\end{bmatrix}$ that also produces the output $\begin{bmatrix} -1\\ 2\end{bmatrix}$.

(d) Is T surjective? Justify your answer.

Solution: The range is spanned by (Theorem SSRLT),

$$\{T(\mathbf{e}_1), T(\mathbf{e}_2), T(\mathbf{e}_3)\} = \left\{\begin{bmatrix} 2\\ -1\end{bmatrix}, \begin{bmatrix} 1\\ 1\end{bmatrix}, \begin{bmatrix} -3\\ 3\end{bmatrix}\right\}$$

This set is linearly dependent (Theorem MVSLD), but we want to go further and determine just what subspace of $\mathbb{C}^2$ it spans. Make the vectors the rows of a matrix and row-reduce,

$$\begin{bmatrix} 2 & -1 \\ 1 & 1 \\ -3 & 3 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 \\ 0 & \boxed{1} \\ 0 & 0 \end{bmatrix}$$

So the range of T is

$$\mathcal{R}(T) = \left\langle \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \right\rangle = \mathbb{C}^2$$

By Theorem RSLT, T is surjective.

(e) Find an element of the codomain, such that there is no element of the domain which when used to evaluate T produces the chosen element of the codomain. Or explain why this is not possible.

Solution: T is surjective, so it is impossible to find such an element.

2. Consider the linear transformation S below, where P_1 is the vector space of polynomials in x with degree 1 or less, and M_{12} is the vector space of 1×2 matrices. (40 points)

$S\colon P_1 \to M_{12}, \quad S(a + bx) = \begin{bmatrix} 2a + 3b & 3a + 4b \end{bmatrix}$

Solution:

(a) Prove that S is invertible, without using any results obtained in the next part.

Solution: A polynomial $a + bx$ is in the kernel of S if $\begin{bmatrix} 2a + 3b & 3a + 4b \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$. The resulting homogeneous system of equationsonly has the trivial solution $a = b = 0$, so the kernel is trivial. Thus by Theorem KILT, S is injective.

Now by Theorem RPNDD,

$$2 = \dim(P_1) = \dim(\mathcal{R}(S)) + \dim(\mathcal{K}(S)) = \dim(\mathcal{R}(S)) + 0 = \dim(\mathcal{R}(S))$$

so $\dim(\mathcal{R}(S)) = 2$. As $\mathcal{R}(S)$ is a subspace of M_{12}, a vector space of dimension 2, we conclude $\mathcal{R}(S) = M_{12}$ (Theorem EDYES). By Theorem RSLT, S is surjective.

Finally, by Theorem ILTIS, S is invertible.

(b) Compute a formula for the inverse of S, S^{-1}.

Solution: By Theorem LTDB it "is enough" to know what S^{-1} does to a basis of M_{12}. A good choice for a basis is $C \left\{ \begin{bmatrix} 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \end{bmatrix} \right\}$. The pre-image of any element of M_{12} will be a set of size 1; we want this element of P_1 for each basis element in C.

$$\begin{bmatrix} 2a + 3b & 3a + 4b \end{bmatrix} = S(a + bx) = \begin{bmatrix} 1 & 0 \end{bmatrix}$$
$$\begin{bmatrix} 2 & 3 & 1 \\ 3 & 4 & 0 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & -4 \\ 0 & \boxed{1} & 3 \end{bmatrix}$$
$$S^{-1}\left(\begin{bmatrix} 1 & 0 \end{bmatrix}\right) = a + bx = -4 + 3x$$

$$\begin{bmatrix} 2a + 3b & 3a + 4b \end{bmatrix} = S(a + bx) = \begin{bmatrix} 0 & 1 \end{bmatrix}$$
$$\begin{bmatrix} 2 & 3 & 0 \\ 3 & 4 & 1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 3 \\ 0 & \boxed{1} & -2 \end{bmatrix}$$
$$S^{-1}\left(\begin{bmatrix} 0 & 1 \end{bmatrix}\right) = a + bx = 3 - 2x$$

So for a general formula we can create

$$\begin{aligned} S^{-1}\left(\begin{bmatrix} p & q \end{bmatrix}\right) &= S^{-1}\left(p \begin{bmatrix} 1 & 0 \end{bmatrix} + q \begin{bmatrix} 0 & 1 \end{bmatrix}\right) \\ &= pS^{-1}\left(\begin{bmatrix} 1 & 0 \end{bmatrix}\right) + qS^{-1}\left(\begin{bmatrix} 0 & 1 \end{bmatrix}\right) \\ &= p(-4 + 3x) + q(3 - 2x) \\ &= (-4p + 3q) + (3p - 2q)x \end{aligned}$$

(c) Fill-in the blank: The work above allows us to conclude that P_1 and M_{12} are ______________ vector spaces.

Solution: "isomorphic"

3. Suppose V is a vector space and $T\colon V \to V$ is a linear transformation. Then a vector $\mathbf{v}$ is an **eigenvector of T** if $T(\mathbf{v}) = \lambda\mathbf{v}$ for some $\lambda \in \mathbb{C}$. Prove that if $\mathbf{v}, \mathbf{u} \in V$ are both eigenvectors of T with the same scalar λ, then $\mathbf{v}+\mathbf{u}$ is also an eigenvector of T. Write a careful proof with explanations for each step. (15 points)

   Solution: Check if $\mathbf{v}+\mathbf{u}$ is an eigenvector of T,

$$\begin{aligned} T(\mathbf{v}+\mathbf{u}) &= T(\mathbf{v}) + T(\mathbf{u}) && \text{Definition LT} \\ &= \lambda\mathbf{v} + \lambda\mathbf{u} && \text{Hypothesis} \\ &= \lambda(\mathbf{v}+\mathbf{u}) && \text{Property DVA} \end{aligned}$$

4. Suppose that $T\colon U \to V$ is a linear transformation, $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_m\}$ is a subset of U and that $C = \{T(\mathbf{u}_1), T(\mathbf{u}_2), T(\mathbf{u}_3), \ldots, T(\mathbf{u}_m)\}$ is a spanning set for V. Prove T is surjective. (15 points)

   Solution: To establish that T is surjective, we will show that every element of V is an output of T for some input (Definition SLT). Suppose that $\mathbf{v} \in V$. As an element of V, we can write $\mathbf{v}$ as a linear combination of the spanning set C. So there are are scalars, $b_1, b_2, b_3, \ldots, b_m$, such that

$$\mathbf{v} = b_1T(\mathbf{u}_1) + b_2T(\mathbf{u}_2) + b_3T(\mathbf{u}_3) + \cdots + b_mT(\mathbf{u}_m)$$

   Now define the vector $\mathbf{u} \in U$ by

$$\mathbf{u} = b_1\mathbf{u}_1 + b_2\mathbf{u}_2 + b_3\mathbf{u}_3 + \cdots + b_m\mathbf{u}_m$$

   Then

$$\begin{aligned} T(\mathbf{u}) &= T(b_1\mathbf{u}_1 + b_2\mathbf{u}_2 + b_3\mathbf{u}_3 + \cdots + b_m\mathbf{u}_m) \\ &= b_1T(\mathbf{u}_1) + b_2T(\mathbf{u}_2) + b_3T(\mathbf{u}_3) + \cdots + b_mT(\mathbf{u}_m) && \text{Theorem LTLC} \\ &= \mathbf{v} \end{aligned}$$

   So, given any choice of a vector $\mathbf{v} \in V$, we can design an input $\mathbf{u} \in U$ to produce $\mathbf{v}$ as an output of T. Thus, by Definition SLT, T is surjective.