Sicherman Dice

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The subject of Sicherman dice began with a question posed and answered by Colonel George Sicherman which was originally reported in the February 1978 issue of *Scientific American*. Sicherman wanted to determine if there was another way to label a pair of six-sided dice, using only positive integers such that the probabilities of the sums would be the same as a standard dice. Sicherman discovered that, in fact, there is one other way. These are the Sicherman dice, one of which is labeled 1, 2, 2, 3, 3, and 4 and the other is labeled 1, 3, 4, 5, 6, and 8. The tables below show the sums of the regular dice and Sicherman dice, respectively. Looking at these tables it can be seen that for the all of the possible sums of the two dice, 2,3,4,...,12, the probabilities are $\frac{1}{36}$, $\frac{2}{36}$, $\frac{3}{36}$, $\frac{4}{36}$, $\frac{5}{36}$, $\frac{6}{36}$, $\frac{5}{36}$, $\frac{4}{36}$, $\frac{3}{36}$, $\frac{2}{36}$, and $\frac{1}{36}$.

+	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

+	1	2	2	3	3	4
1	2	3	3	4	4	5
3	4	5	5	6	6	7
4	5	6	6	7	7	8
5	6	7	7	8	8	9
6	7	8	8	9	9	10
8	9	10	10	11	11	12

Although it is trivial to see that Sicherman dice do, in fact, provide the same probability of rolling each sum as standard dice, it is not as trivial a matter to find the labels for the Sicherman dice in the first place. Additionally, it is not trivial that these relabelings are the only other possible ways to label two six-sided dice to achieve the desired sum probabilities. There are several different approaches to deriving the Sicherman dice labelings, three of which will be explored. Two of these will be theoretical approaches and one will be a programmatic approach. The discussion of six-sided dice will also be generalized into a discussion of k and l-sided dice with the same sum probabilities. Furthermore, the discussion of dice labelings can be expanded to a discussion of the faces of k dice formed from the Platonic solids and cyclotomic polynomials.

Programmatic Solution to Dice Labels

An interesting way to test the number of dice labelings for two six-sided dice that result in the desired probabilities as well as determine those labels is to create a program. By writing a simple algorithm, we can utilize the speed and computational abilities of a computer to perform a task that does not require much thought, but would take far too long to ever do by hand. It is relatively straightforward to create a program that will generate all of the possible dice labels that could potentially result in the standard probability of sums. Using some reasonable deduction and simple number theory, you can reduce the number of possibilities. The smallest sum is 2 and it only occurs one time, so 1 must occur exactly one time on each of the die, because we are restricted to using positive integers. We also know that the sum 3 occurs twice and each die has a 1, so at least one of the dice must have a 2. Now that at least one dice has a two, we can make 10 the highest possible value. We could limit the values further, but this gives us a start and limits the amount of time that the program will run. We can then create all of the possible combinations of pairs of six integers in our range and add all of their sums. We then compare those sums to the desired sums and if a given choice matches the desired probability, we sort it and add it to a list. If the dice labelings that we add to the list are sorted and we check the list for repeats, then at the end of the program, the list will contain exactly the labelings for two six-sided dice that give the normal probabilities. When this was implemented, the final product was, as expected, one pair of dice with the standard labels and one pair with the labels $\{1, 2, 2, 3, 3, 4\}$ and $\{1, 3, 4, 5, 6, 8\}$. Although this method does not give any insight into the cause of exactly two dice labelings, it does give a basis to further explore the problem and proves that the result that we are trying to find is true.

Dice Labels and Unique Factorization Domains

Now that we have shown concretely that there are exactly two labelings with the desired probabilities, the next step is to discuss the theory behind our result. Before we get to the dice, we must first show that every nonconstant polynomial in $\mathbf{Z}[x]$ can be written in the form $cp_i(x)p_2(x)\cdots p_k(x)$ where $c \in \mathbf{Z}$ and each $p_i(x)$ is primitive and irreducible over $\mathbf{Z}$. To show this, let *f(x)* be a nonconstant polynomial in $\mathbf{Z}[x]$. To show that there is a factorization, we will use induction on the degree of $f(x)$. If the degree of $f(x)$ is 1, then $f(x)$ is irreducible and we are done. Assume that every polynomial of degree less than that of $f(x)$ can be written as a product of irreducibles. If $f(x)$ is irreducible, we are done. If not, $f(x) = g(x)h(x)$ where both $g(x)$ and $h(x)$ have degree less than that of $f(x)$. By the induction hypothesis, both $g(x)$ and $h(x)$ can be written as the product of irreducibles. Therefore, $f(x)$ is also a product of irreducibles. It must now be shown the uniqueness of the factorization. To do this, suppose a nonconstant polynomial $f(x)$ has two factorizations into irreducibles. So, $f(x) = cp_1(x)\cdots p_k(x) = bq_1(x)\cdots q_t(x)$ where $c, b \in \mathbf{Z}$ and the *p(x)*'s and *q(x)*'s are primitive. Therefore, $p_1(x)\cdots p_k(x)$ and $q_1(x)\cdots q_t(x)$ are also primitive and so c and b must equal plus-or-minus the content of $f(x)$, meaning they are equal in absolute value. If we view the *p(x)*'s and *q(x)*'s as elements of $\mathbf{Q}[x]$ and noting that $p_1(x)$ divides $q_1(x)\cdots q_t(x)$ then $p_1(x)|q_i(x)$ for some i. Then by renumbering, we are able to assume that $i = 1$. Since $q_1(x)$ is irreducible, we get $q_1(x) = (r/s)p_1(x)$, where $r, s \in \mathbf{Z}$. Both $q_i(x)$ and $p_1(x)$ are primitive, however, so we must have $r/s = \pm 1$. This gives us $q_1(x) = \pm p_1(x)$. After cancelling, we have $p_2(x)\cdots p_k(x) = \pm q_2(x)\cdots q_t(x)$. We repeat the argument with $p_2(x)$ instead of $p_1(x)$. If $k < t$, after k repititions of the argument we have 1 on the left and a nonconstant polynomial on the right. This is impossible and if $k\,t$, after t steps we would have ± 1 on the right and a nonconstant polynomial on the left. Again, this is impossible, which leads to the conclusion that $k = t$ and $p_i(x) = \pm q_i(x)$ after renumbering the q_i's. We have proven our result and can now use the fact of unique factorization in $\mathbf{Z}[x]$ to solve the Sicherman dice problem. [1]

The fact that there is only one possible relabeling for a pair of six-sided dice that yields the same probability as regular dice using only positive integers is due to the fact that $\mathbf{Z}$[x] is a unique factorization domain. To see this, consider the product of polynomials for a pair of standard dice. These are created using the dice labels as exponents. Doing this, we get $(x^6 + x^5 + x^4 + x^3 + x^2 + x)(x^6 + x^5 + x^4 + x^3 + x^2 + x) =$

$$x^2 + 2x^3 + 3x^4 + 4x^5 + 5x^6 + 6x^7 + 5x^8 + 4x^9 + 3x^{10} + 2x^{11} + x^{12}.$$

From this polynomial, you can see that the coefficients correspond to the number of ways each sum can be achieved. The correspondence between pairs of labels whose sum is any certain number, for example 6, and pairs of terms whose product is x^6 i s one-to-one. This result is valid for any possible sum as well as any other dice, including Sicherman dice, that give the same probability. [1]

Assume that there are two dice labeled with positive integers that give us the same probabilities as regular dice and let $\{a_1, a_2, a_3, a_4, a_5, a_6\}$ and $\{b_1, b_2, b_3, b_4, b_5, b_6\}$ be the sets of these labels. Based on what we know about the product of polynomials this means that $(x^6 + x^5 + x^4 + x^3 + x^2 + x)(x^6 + x^5 + x^4 + x^3 + x^2 + x) = (x^{a_1} + x^{a_2} + x^{a_3} + x^{a_4} + x^{a_5} + x^{a_6})(x^{b_1} + x^{b_2} + x^{b_3} + x^{b_4} + x^{b_5} + x^{b_6})$. To find the labels for the dice, we just need to solve for the a_i and b_i, with $1 \leq i \leq 6$. Because $\mathbf{Z}$[x] is a unique factorization domain, we can show that $x^6 + x^5 + x^4 + x^3 + x^2 + x$ factors uniquely into irreducibles as $x(x+1)(x^2+x+1)(x^2-x+1)$. This gives us that $(x^6 + x^5 + x^4 + x^3 + x^2 + x)(x^6 + x^5 + x^4 + x^3 + x^2 + x) = x^2(x+1)^2(x^2+x+1)^2(x^2-x+1)^2$. By theorem, we know that these are the only possible irreducible factors of $P(x) = x^{a_1} + x^{a_2} + x^{a_3} + x^{a_4} + x^{a_5} + x^{a_6}$, so *P(x)* must have the form $x^q(x+1)^r(x^2+x+1)^s(x^2-x+1)^t$ where $0 \leq q, r, s, t \leq 2$.

We can reduce the choices for parameters by some simple deduction. First, evaluating $P(1)$ will restrict the possibilities for r and s. $P(1) = 1^{a_1} + 1^{a_2} + 1^{a_3} + 1^{a_4} + 1^{a_5} + 1^{a_6} = 6 = 1^q 2^r 3^s 1^t$. From this r and s must equal 1. Now evaluating $P(0)$ both ways, we get $P(0) = 0 = 0^q 1^1 1^1 1^t$, so q cannot equal 0. If we let $q = 2$, then the smallest possible sum with another dice is 3, because $x^2(x+1)(x^2+x+1)(x^2-x+1)^0 = x^5 + 2x^4 + 2x^3 + x^2$. This gives the set of labels $\{5, 4, 4, 3, 3, 2\}$ and because we are restricted to using positive integers, a sum 2 can never be achieved, violating the assumption that these dice will have the same sums as regular dice. Our only option, then, is to let $q = 1$. Knowing that $q = 1$, $r = 1$, and $s = 1$ just leaves the options for t being 0,1, or 2. Going through the values of u in turn, we see that when $u = 0$, $P(x) = x^4 + x^3 + x^3 + x^2 + x^2 + x$ which gives the die labels $\{4, 3, 3, 2, 2, 1\}$. When $u = 1$, $P(x) = x^6 + x^5 + x^4 + x^3 + x^2 + x$. So the die labels for $u = 1$ are $\{6, 5, 4, 3, 2, 1\}$, which is a regular dice. Finally, $u = 2$ gives $P(x) = x^8 + x^6 + x^5 + x^4 + x^3 + x$ which has die labels $\{8, 6, 5, 4, 3, 1\}$. So, the die labels provided when $u = 0$ and $u = 2$ are the labels for our second set of dice. Verifying that we get the desired polynomial product, we multiply $x^6 + x^5 + x^4 + x^3 + x^2 + x$ and $x^8 + x^6 + x^5 + x^4 + x^3 + x$,

$$(x^6 + x^5 + x^4 + x^3 + x^2 + x)(x^8 + x^6 + x^5 + x^4 + x^3 + x) =$$

$$(x^2 + 2x^3 + 3x^4 + 4x^5 + 5x^6 + 6x^7 + 5x^8 + 4x^9 + 3x^{10} + 2x^{11} + x^{12}).$$

So, the new dice labels yield the desired results and, of course, $\{1, 2, 2, 3, 3, 4\}$ and $\{1, 3, 4, 5, 6, 8\}$ are the labels for Sicherman dice and we have shown that this is the only possible relabeling giving the same sum probabilities. [1]

Dice Labels and the Probability Generating Function

Now we will employ a different method to determine dice relabelings with the desired probabilities. To do this, we will use the probability generating function. If X is a discrete random variable, the probability generating function of x is $\psi(t) = E(t^X) = \sum_x p(x)t^x$ for $0 \le t \le 1$, where the sum is taken over all of the possible outcomes of x in the given sample space, and $p(x) = P(X = x)$. One interesting and useful property of probability generating functions is given from the notion of independence. If X and Y are independent random variables with probability functions $\psi_x(t)$ and $\psi_y(t)$, respectively, then the sum $Z = X + Y$ has the probability generating function $\psi_z(t) = \psi_x(t)\psi_y(t)$. Two dice behave independently of one another, so the probability generating function of a sum from two six-sided dice is $\psi(t) = \frac{1}{6}(t^{a_1} + t^{a_2} + t^{a_3} + t^{a_4} + t^{a_5} + t^{a_6}) * \frac{1}{6}(t^{b_1} + t^{b_2} + t^{b_3} + t^{b_4} + t^{b_5} + t^{b_6}) = \sum_{i=1}^{6}\sum_{j=1}^{6} \frac{1}{36} t^{a_i+b_j}$, where the labels on the 6 faces of the first die are $a_1, a_2, ..., a_6$ and the second die is labeled $b_1, b_2, ..., b_6$. The sums for the dice have the probability generating function, $\psi(t) = (\frac{1}{6}(t + t^2 + t^3 + t^4 + t^5 + t^6))^2 = \frac{1}{36}(t^2 + 2t^3 + 3t^4 + 4t^5 + 5t^6 + 6t^7 + 5t^8 + 4t^9 + 3t^{10} + 2t^{11} + t^{12})$. This probability generating function is what is used to relabel a pair of six-sided dice. From here, the analysis is much the same as that of the previous section, where the probability generating function is expressed as a product of irreducible factors, $\psi(t) = (\frac{1}{6}(1 + t)(1 - t + t^2)(1 + t + t^2))^2 = \frac{t^2}{36}(1 + t)^2(1 - t + t^2)^2(1 + t + t^2)^2$. Then to get the Sicherman labels, let $\phi_1(t) = \frac{1}{6}(1 + t)(1 - t + t^2)^2(1 + t + t^2)$ and $\phi_2(t) = \frac{1}{6}(1 + t)(1 + t + t^2)$. Multiplying these together, it is seen that $\phi_1(t)\phi_2(t) = \psi(t)$ and expanding $\phi_1(t)$ and $\phi_2(t)$ give $\phi_1(t) = \frac{1}{6}(t + t^3 + t^4 + t^5 + t^6 + t^8)$ and $\phi_2(t) = \frac{1}{6}(t + 2t^2 + 2t^3 + t^4)$, which once again gives us the Sicherman dice labelings. [2]

Expanding the Dice Problem

Now that we have proven the existence of exactly two dice labelings that give the standard sums and probabilities, we can generalize this problem to explore if we can find a k-sided die together with an l-sided die and get the same sums with the same probabilities. Once again, probability generating functions will be used to answer the problem. Obviously, $k \times l = 36$, which limits the choices for k and l to $\{2, 18\}, \{3, 12\}, \{4, 9\}$, and $\{6, 6\}$. The latter of which we have already solved. As an example, the probability generating function $\phi_1(t) = \frac{t}{2}(1 + t) = \frac{1}{2}(t + t^2)$ represents a die with labels $\{1, 1, 1, 2, 2, 2\}$. This die can be generalized as an equally weighted coin, or in the terminology of the problem as a two-sided dice. From this, we know that we need to pair it with an 18-sided dice to obtain a product of 36. The 18-sided dice comes from the probability generating function giving $\phi_2(t) = \frac{t}{18}(1+t)(1-t+t^2)^2(1+t+t^2)^2 = \frac{1}{18}(t+t^2+2t^3+2t^4+3t^5+3t^6+2t^7+2t^8+t^9+t^10)$, which means that it has labels $\{1, 2, 3, 3, 4, 4, 5, 5, 5, 6, 6, 6, 7, 7, 8, 8, 9, 10\}$. Another combination that gives an 18-sided die and a two-sided die come from the probability generating functions $\frac{1}{18}(1 + t)(1 - t + t^2)(1 + t + t^2)^2 = \frac{1}{18}(t + 2t^2 + 3t^3 + 3t^4 + 3t^5 + 3t^6 + 2t^7 + t^8)$ and

$\frac{t}{2}(1+t)(1-t-t^2) = \frac{t+t^4}{2}$. These result in labels $\{1, 2, 2, 3, 3, 3, 4, 4, 4, 5, 5, 5, 6, 6, 6, 7, 7, 8\}$ and $\{1, 4\}$, respectively. One set of labels for a three-sided and twelve-sided pair of dice is $\{1, 2, 3\}$ and $\{1, 2, 3, 4, 4, 5, 5, 6, 6, 7, 8, 9\}$. These labels come from $\phi_1(t) = \frac{t}{3}(1+t+t^2) = \frac{1}{3}(t+t^2+t^3)$ and $\phi_2(t) = \frac{t}{12}(1+t)^2(1-t+t^2)^2(1+t+t^2) = \frac{1}{12}(t+t^2+t^3+2t^4+2t^5+2t^6+t^7+t^8+t^9)$. Additionally, one set of labels for a four-sided and nine-sided pair of dice is $\{1, 2, 2, 3\}$ and $\{1, 3, 3, 5, 5, 5, 7, 7, 9\}$. These labels come from $\phi_1(t) = \frac{t}{4}(1+t)^2 = \frac{1}{4}(t+2t^2+t^3)$ and $\phi_2(t) = \frac{t}{9}(1-t+t^2)^2(1+t+t^2)^2 = \frac{1}{9}(t+2t^3+3t^5+2t^2+9)$. There are other labeling possibilities giving the desired probabilities, but this at least gives an example dice labeling for each of the possible dice sizes. [3]

Cyclotomic Polynomials

Cyclotomic polynomials play an important role in analyzing dice labelings of six-sided dice as well as other sizes of dice. Let n be a positive integer. The equation $x^n - 1 = 0$ has n complex roots. These roots are called nth roots of unity. Assume ρ is an nth root of unity and let m be the smallest of positive integers j such that $\rho^j = 1$. Then m is the order of ρ in the multiplicative group of nth roots of unity. We know by Lagrange's Theorem that m must divide n. An nth root of unity with order n is a primitive nth root of unity. The polynomial $F_n(x) = \prod(x-\omega)$, as ω goes through the primitive nth roots of unity, is call the nth cyclotomic polynomial. The degree of the function is given by the Euler phi-function, $\phi(n)$, which is the number of integers i where $1 \leq i \leq n$, relatively prime to n. We know that each root of unity is a primitive dth root of unity for some d, where d divides n. This gives $x^n - 1 = \prod_{d|n} F_d(x)$. Additionally, it is irreducible over $\mathbf{Q}$, which will be proven. Furthermore, it is interesting to note that for a prime p, $F_p(x) = 1 + x + \ldots + x^{p-2} + x^{p-1}$, because each primitive pth root of unity is a root of $\frac{(x^p-1)}{(x-1)}$. Next we want to show that the nth cyclotomic polynomial has coefficients from the integers and is irreducible over the rationals. [4]

To show that the nth cyclotomic polynomial is irreducible, we must first prove another result. If $f(x)$ is a monic polynomial and $f(x) = g(x)h(x)\epsilon\mathbf{Q}[x]$ when $g(x)$ and $h(x)$ are monic, then $g(x)$ and $h(x)$ are in $\mathbf{Z}[x]$. To show this, let a and b be the smallest positive integers such that $ag(x) = m(x)$ and $bh(x) = n(x)$ are in $\mathbf{Z}$[x]. This gives $abf(x) = m(x)n(x)$. Now let $m(x) = \sum_{i=0}^{r} a_i x^i$ and $n(x) = \sum_{i=0}^{s} b_i x^i$. The gcd$(a_0, a_1, ..., a_r)$=gcd$(b_0, b_1, ..., b_s)$=1. If $ab = 1$, then we are done. Otherwise, let p be a prime such that p divides ab. Let j and k be the smallest indices such that p does not divide a_i and b_j, respectively. So, p must divide the coefficient $\ldots + a_{i-2}b_{j+2} + a_{i-1}b_{j+1} + a_i b_j + a_{i+1}b_{j-1} + \ldots$ of x^{i+j}, and it divides every term except $a_i b_j$. So $ab = 1$ and we have proven our result. [5]

Now, to prove that the nth cyclotomic polynomial is irreducible over $\mathbf{Q}$ with coefficients from $\mathbf{Z}$, let ω be a primitive nth root of 1, and let $f(x)$ be its minimum polynomial over $\mathbf{Q}$. Let p be a prime not dividing n, and let $g(x)$ be the minimum polynomial over $\mathbf{Q}$ of the primitive nth root ω^p. By the previous result, we know that $f(x)$ and $g(x)$ are in $\mathbf{Z}$[x]. If $f(x) \neq g(x)$, then they are relatively prime in $\mathbf{Q}[x]$, and therefore have no common roots. They both divide $x^n - 1 = f(x)g(x)h(x)$, with $h(x) \in \mathbf{Z}[]$ as well. The polynomial $g(x^p)$ has ω as a root, from $g(x^p) = f(x)g(x)$, with $k(x) \in \mathbf{Z}[]$. Now take the equalities $x^n - 1 = f(x)g(x)h(x)$ and $g(x^p) = f(x)k(x)$ mod p. So, the polynomials are in $(\mathbf{Z}/p\mathbf{Z})[]$

and operating mod p we have $g(x^p) = g(x)^p = f(x)k(x)$, and any prime factor $q(x)$ of $f(x)$ must divide $g(x)$. So $q(x)^2$ must divide $x^n - 1 = f(x)g(x)h(x)$, which means that $x^n - 1$ has multiple roots. However, $x^n - 1$ and nx^{n-1} are relatively prime because p does not divide n. That gives us that in $\mathbf{Q}[x]$, $f(x) = g(x)$. Now let ω^m be any primitive nth root of 1. Then $m = p_1 p_2 p_3 \cdots p_t$, with p_i prime and relatively prime to n. By the previous argument, the minimum polynomial of ω^{p_1} is $f(x)$. For the same reason, the minimum polynomial of $\omega^{p_1 p_2}$ is also $f(x)$. It follows that $f(x)$ is the minimum polynomial over $\mathbf{Q}$ of all of the primitive nth roots of 1. Therefore, $f(x)$ must divide $\phi_n(x)$ in $\mathbf{Z}[]$. However, the degree of $f(x)$ is at least as great as the degree of $\phi_n(x)$, so $\phi_n(x) = f(x)$ and we have proven that the nth cyclotomic polynomial is irreducible over $\mathbf{Q}[x]$. [5]

Now that we have an introduction to cyclotomic polynomials, we can use them to analyze k dice formed from the Platonic solids. We can generalize the labels of a die with f faces as equivalent to an urn containing f balls with the same labels. This allows the analysis to ignore the geometry of the die, which is inconsequential to the problem. We will define the standard labelings of the balls in the urn to be $1, 2, ..., f-1, f$, inclusive, for a set of f balls in an urn. We can characterize the sets of urns containing n balls with the same sum probabilities as a set of standard n-urns, which have the associated polynomial $\sum_{i=1}^{n} x^i = (\frac{x^n-1}{x-1})x$. Let $A_1, ..., A_k$ be urns each containing n balls, labeled with positive integers and $B_1, ...B_r$ be r standard n-urns. The probability that when one ball is chosen from each of $A_1, ..., A_k$, the sum of the labels on the balls that are chosen is t, will be called$P_A[S = t]$. $P_B[S = t]$ is defined similarly for the urns $B_1, ...B_r$. If $P_A[S = t] = P_B[S = t]$, then $r = k$. If each urn contains exactly one ball, the result is trivial. Assume $n > 1$. Since the balls in the urns are labelled with positive integers, $P_A[S = t] \neq 0$ only when $t \geq k$. If $P_A[S = t] \neq 0$ then$P_A[S = t] \geq \frac{1}{n^k}$, since there are n^k ways to choose one ball from each of $A_1, ..., A_k$. Since $P_A[S = t] = P_B[S = t] = \frac{1}{n^r} \neq 0$, we have $k \leq r$ from the first condition and $\frac{1}{n^r} \geq \frac{1}{n^k}$ from the second condition. Therefore, $r = k$, as is required. Now assume that $P_A[S = t] = P_B[S = t]$ for all t. Allow $A_i(x)$ to be the polynomial associated with urn A_i, where the coefficient of x^j in $A_i(x)$ is the number of balls labelled j in urn A_i. We want to show the following: $A_i(0) = 0$, $A_i(1) = n$; all of the coefficients of $A_i(x)$ are non-negative integers; the coefficient of x in $A_i(x)$ is one; and, $\prod_{i=1}^{k} A_i(x) = (\frac{(x^n-1)x}{x-1})^k$. Additionally, if the previous conditions are true for a set of k urns, then it has the same sum of probabilities as a set of k standard n-urns. [6]

From the definition of $A_i(x)$, it is clear that if $P_A[S = t] = P_B[S = t]$ for all t, then $A_i(0) = 0$, $A_i(1) = n$ and all of the coefficients of $A_i(x)$ are non-negative integers, which satisfies the confines of the problem. The number of ways to choose one ball from each of the A_i and get a sum of t is the same as of coefficient of x^t of $\prod_{i=1}^{k} A_i(x)$. So, $\prod_{i=1}^{k} A_i(x) = n^k \sum_k P_A[S = t]x^t$. Likewise, $[(x^n - 1)x/(x - 1)]^r = n^r \sum_k P_B[S = t]x^t$. We have already showed that $r = k$, so we have established that $\prod_{i=1}^{k} A_i(x) = (\frac{(x^n-1)x}{x-1})^k$. If we compare terms of lowest degree in $\prod_{i=1}^{k} A_i(x) = (\frac{(x^n-1)x}{x-1})^k$ and recall that $A_1(x), ..., A_k(x)$ have no constant terms, it follows that the coefficient of x in $A_i(x)$ is one.

We can now use cyclotomic polynomials to discuss D, an n-faced die which can be combined with other n-faced dice to form a set of k dice which has the same sum probabilities as a set of standard n-faced dice. As previously shown, the associated polynomial *D(x)* is a factor of $[(x^n - 1)x/(x - 1)]^k$. From the discussion on cyclotomic polynomials, we know that $(s^n - 1)/(x - 1)$ is the product of the dth cyclotomic polynomials where d ranges over the divisors of n with $d > 1$. Additionally, we know that the cyclotomic polynomials are irreducible over the rationals. Therefore, *D(x)* is of the form $x \prod_{d|n, d>1} F_d(x)^{a_d}$, where $F_d(x)$ is the dth cyclotomic polynomial, a_d is a non-negative integer, and the product is taken over all integers $d > 1$ which divide n. [6]

The subject of Sicherman dice is both interesting in its direct connection to dice and games, but in its application to abstract algebra as well. The issue of the Sicherman labelings, their relationship to unique factorization domains, and the generalization of dice labelings into broader problems creates a great amount of depth to the subject. By slightly changing the dice problem, there are never-ending variations to explore and results to find.

References

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