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Exam 2 Spring 2012
Chapter V

Show *all* of your work and *explain* your answers fully. There is a total of 100 possible points.
Use Sage only to row-reduce matrices and include these computations in your answers.

1. Determine if the vector $\mathbf{y}$ is in the span of the set S, $\langle S\rangle$. (15 points)

$$\mathbf{y}=\begin{bmatrix}2\\-11\\1\\7\end{bmatrix}\qquad S=\{\mathbf{v}_1,\,\mathbf{v}_2,\,\mathbf{v}_3,\,\mathbf{v}_4,\,\mathbf{v}_5,\,\mathbf{v}_6\}=\left\{\begin{bmatrix}1\\0\\-2\\0\end{bmatrix},\begin{bmatrix}-1\\1\\1\\-1\end{bmatrix},\begin{bmatrix}1\\-5\\3\\5\end{bmatrix},\begin{bmatrix}-1\\-4\\3\\2\end{bmatrix},\begin{bmatrix}2\\4\\-5\\-2\end{bmatrix},\begin{bmatrix}4\\6\\5\\-4\end{bmatrix}\right\}$$

$a_1\underset{\sim}{v_1}+a_2v_2+a_3\underset{\sim}{v_3}+a_4\underset{\sim}{v_4}+a_5\underset{\sim}{v_5}+a_6\underset{\sim}{v_6}=y \xrightarrow{\text{SLSLC}}$ system with augmented matrix $\xrightarrow{\text{RREF}}$

$$\left[\begin{array}{cccccc:c}1&0&-4&0&1&0&1\\0&1&-5&0&0&0&-3\\0&0&0&1&-1&0&2\\0&0&0&0&0&1&0\end{array}\right]$$

System is consistent (RCLS) so a_i exist and $\underset{\sim}{y}\in\langle S\rangle$.

2. Determine if the sets of vectors below are linearly independent or not. Be sure to provide sufficient justification. (20 points)

Apply Theorem LIVRN in each part. Vectors become columns of matrix which is row-reduced.

(a) $\left\{\begin{bmatrix}1\\0\\1\\-1\\1\end{bmatrix},\begin{bmatrix}2\\1\\0\\-1\\-3\end{bmatrix},\begin{bmatrix}7\\2\\4\\0\\1\end{bmatrix}\right\}$

matrix $\xrightarrow{\text{RREF}}$ $\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\\0&0&0\\0&0&0\end{bmatrix}$ $3=n=r=3$ so set is linearly independent

(b) $\left\{\begin{bmatrix}1\\-2\\0\\0\\1\end{bmatrix},\begin{bmatrix}-6\\1\\-3\\1\\2\end{bmatrix},\begin{bmatrix}-8\\5\\-3\\1\\0\end{bmatrix}\right\}$

matrix $\xrightarrow{\text{RREF}}$ $\begin{bmatrix}1&0&-2\\0&1&1\\0&0&0\\0&0&0\\0&0&0\end{bmatrix}$ $3=n\neq r=2$ so the set is linearly dependent

3. The set S below is the same as in Question 1. Find a linearly independent set T so that $\langle T\rangle = \langle S\rangle$. (10 points)

$$S = \{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3,\ \mathbf{v}_4,\ \mathbf{v}_5,\ \mathbf{v}_6\} = \left\{ \begin{bmatrix}1\\0\\-2\\0\end{bmatrix}, \begin{bmatrix}-1\\1\\1\\-1\end{bmatrix}, \begin{bmatrix}1\\-5\\3\\5\end{bmatrix}, \begin{bmatrix}-1\\-4\\3\\2\end{bmatrix}, \begin{bmatrix}2\\4\\-5\\-2\end{bmatrix}, \begin{bmatrix}4\\6\\5\\-4\end{bmatrix} \right\}$$

By theorem BS we just collect v_1, v_2, v_4 since $D = \{1, 2, 4, 6\}$

$T = \{v_1, v_2, v_4, v_6\}$

4. The vector $\mathbf{y}$ below is the same as in Question 1. Find a linear combination of the vectors in the set set T (that you found in the previous question) that equals $\mathbf{y}$. Comment thoughtfully on the relationship between the results in Question 1, the previous question, and this question. (10 points)

$$\mathbf{y} = \begin{bmatrix}2\\-11\\1\\7\end{bmatrix}$$

Form linear combination, use SLSLC to form a system and row-reduce augmented matrix

$$[v_1|v_2|v_4|v_6|y] \xrightarrow{\text{RREF}} \begin{bmatrix}1&0&0&0&1\\0&1&0&0&-3\\0&0&1&0&2\\0&0&0&1&0\end{bmatrix}$$

$y = 1\,v_1 + (-3)v_2 + 2v_4$

there are many linear combos of S that equal y. There is just one linear combo of T that equals y.

5. Find a linearly independent set R whose span is the null space of the matrix A below. In other words, R will be linearly independent and $\langle R\rangle = \mathcal{N}(A)$. (10 points)

$$\begin{bmatrix}1&-3&7&-4&4\\1&-2&3&-2&1\\0&2&-8&5&-8\\0&-1&4&2&-5\end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix}1&0&-5&0&-1\\0&1&-4&0&1\\0&0&0&1&-2\\0&0&0&0&0\end{bmatrix}$$

$F = \{3, 5\}$

Apply theorem BNS

$$N(A) = \left\langle\left\{\begin{bmatrix}\_\\\_\\1\\\_\\0\end{bmatrix}, \begin{bmatrix}\_\\\_\\0\\\_\\1\end{bmatrix}\right\}\right\rangle = \left\langle\left\{\begin{bmatrix}5\\4\\1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\-1\\6\\2\\1\end{bmatrix}\right\}\right\rangle$$

6. Given two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$ define a new operation, called *subtraction*, by $[\mathbf{u} - \mathbf{v}]_i = [\mathbf{u}]_i - [\mathbf{v}]_i$, $1 \leq i \leq m$. Prove that subtraction is not really anything new (because we can accomplish subtraction with operations we already have) by showing that $\mathbf{u} - \mathbf{v} = \mathbf{u} + (-1)\mathbf{v}$. (10 points)

For $1 \leq i \leq m$

$[u - v]_i = [u]_i - [v]_i$ Definition

$= [u]_i + (-1)[v]_i$ scalar operation

$= [u]_i + [(-1)v]_i$ CVSM

$= [u + (-1) v]_i$ CVA

So by Defn CVE $u - v = u + (-1) v$

7. Referring to the result about subtraction from the previous question, prove that for $\alpha \in \mathbb{C}$ and $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$, $\alpha(\mathbf{u} - \mathbf{v}) = \alpha\mathbf{u} - \alpha\mathbf{v}$. (10 points)

$\alpha(u - v) = \alpha(u + (-1) v)$ Problem 6

$= \alpha u + \alpha((-1) v)$ DVAC

$= \alpha u + (-1)(\alpha v)$ SMAC, scalar commutativity

$= \alpha u - \alpha v$ Problem 6

Notice we do not need to use $[\ ]_i$ & Defn CVE.

8. Suppose that $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{C}^m$. Prove that $\langle\{\mathbf{v}_1, \mathbf{v}_2\}\rangle = \langle\{\mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_1 - \mathbf{v}_2\}\rangle$. (10 points)

By Defn SE, need to prove two subset statements

1) $\langle\{v_1 + v_2, v_1 - v_2\}\rangle \subseteq \langle\{v_1, v_2\}\rangle$?

$v_1 + v_2, v_1 - v_2 \in \langle\{v_1, v_2\}\rangle$ is enough to show this

2) $\langle\{v_1, v_2\}\rangle \subseteq \langle\{v_1 + v_2, v_1 - v_2\}\rangle$

$v_1 = \frac{1}{2}(v_1 + v_2) + \frac{1}{2}(v_1 - v_2)$

$v_2 = \frac{1}{2}(v_1 + v_2) - \frac{1}{2}(v_1 - v_2)$

is enough to show this.