Math 290 Name: Key Dr. Beezer
Exam 3 Spring 2012
Chapter M

Show *all* of your work and *explain* your answers fully. There is a total of 100 possible points.
Use Sage only to row-reduce matrices or to compute extended echelon form, except where indicated in a problem statement. Include the results of these computations in your answers and describe the input used.

1. Other than actually asking Sage to compute the inverse of the matrix A below, what Sage command could you use to see that A has an inverse? Then compute the inverse of A only using just the reduced row-echelon form command, `.rref()`, on an appropriate matrix. (You can create the "appropriate" matrix with any Sage commands you like.) (15 points)

$$A = \begin{bmatrix} -2 & -3 & -1 & 0 \\ 3 & 4 & 2 & -2 \\ -3 & -4 & -1 & 0 \\ -4 & -6 & -4 & 5 \end{bmatrix}$$

`not A.is_singular()`
`A.is_invertible()`
} two good choices

`A.inverse()` gives

$$A^{-1} = \begin{bmatrix} 0 & 5 & 2 & 2 \\ 1 & -5 & -3 & -2 \\ -4 & 5 & 5 & 2 \\ -2 & 2 & 2 & 1 \end{bmatrix}$$

2. Determine a linearly independent spanning set for the column space of B in two different ways, meeting the requirements given. (20 points)

$$B = \begin{bmatrix} 0 & -1 & -1 & 2 & -1 \\ -2 & 7 & 3 & -16 & 13 \\ 3 & -10 & -4 & 23 & -19 \\ -3 & 10 & 4 & -23 & 19 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 2 & 1 & -3 \\ 0 & 1 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) The set contains only vectors that are columns of B.

$D = \{1, 2\}$ "pivot columns" By Theorem BCS:

$$C(B) = \left\langle \left\{ \begin{bmatrix} 0 \\ -3 \\ 3 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ 7 \\ -10 \\ 10 \end{bmatrix} \right\} \right\rangle$$

(b) The set should be obtained in the most computationally efficient manner possible.

Use rref: $C(B) = R(A^t) \xrightarrow{\text{rref}} R(B)$ but throw out zero rows of B

$$A^t \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -1/2 & 1/2 \\ 0 & 1 & -3/2 & 3/2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad C(B) = \left\langle \left\{ \begin{bmatrix} 1 \\ 0 \\ -1/2 \\ 1/2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -3/2 \\ 3/2 \end{bmatrix} \right\} \right\rangle$$

3. Consider the matrix B from the previous question. (20 points)

(a) Use the matrix L from the extended echelon form to compute a linearly independent set whose span equals the column space of B.

$$[B \mid I_4] \xrightarrow{\text{RREF}} \left[\begin{array}{ccccc|cccc} 1 & 0 & 2 & 1 & -3 & 0 & 10 & 0 & -7 \\ 0 & 1 & 1 & -2 & 1 & 0 & 3 & 0 & -2 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & 3 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right] \leftarrow L$$

$$C(B) = N(L) = \left\langle \left\{ \begin{bmatrix} \_ \\ 1 \\ \_ \\ 0 \end{bmatrix}, \begin{bmatrix} \_ \\ 0 \\ \_ \\ 1 \end{bmatrix} \right\} \right\rangle = \left\langle \left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\} \right\rangle$$

(b) Use the matrix L from the extended echelon form to compute a linearly independent set whose span equals the left null space of B.

$$\mathcal{L}(B) = \mathcal{R}(L) = \left\langle \left\{ \begin{bmatrix} 1 \\ 3 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\} \right\rangle$$

non zero rows of L,
written as columns

4. Suppose that A and B are both $m \times n$ matrices. Prove that $(A+B)^t = A^t + B^t$. (15 points)

For $1 \le i \le m$, $1 \le j \le n$

$$\begin{aligned} [(A+B)^t]_{ij} &= [A+B]_{ji} && \text{TM} \\ &= [A]_{ji} + [B]_{ji} && \text{MA} \\ &= [A^t]_{ij} + [B^t]_{ij} && \text{TM} \\ &= [A^t + B^t]_{ij} && \text{MA} \end{aligned}$$

So by Definition ME, $(A+B)^t = A^t + B^t$

5. When computing the extended echelon form of an $m \times n$ matrix A we compute $[A|I_m] \xrightarrow{\text{RREF}} [B|J]$. Prove that J is nonsingular. (15 points)

① See the first paragraph of the proof of theorem PEEF.

② Is J row-equivalent to Im? What is the sequence of row operations? We need to know that the row operations which convert Im to J can be <u>reversed</u> to create J to Im. See Exercise RREF.T10.

6. Suppose $\alpha \in \mathbb{C}$ is a scalar, A is an $m \times n$ matrix and B is an $n \times p$ matrix. Prove that $(\alpha A)B = A(\alpha B)$. (15 points)

For $1 \le i \le m$, $1 \le j \le p$,

$$
\begin{aligned}
[(\alpha A)B]_{ij} &= \sum_{k=1}^{n} [\alpha A]_{ik} [B]_{kj} && \text{EMP} \\
&= \sum_{k=1}^{n} \alpha [A]_{ik} [B]_{kj} && \text{MSM} \\
&= \sum_{k=1}^{n} [A]_{ik} (\alpha [B]_{kj}) && \text{CMCM} \\
&= \sum_{k=1}^{n} [A]_{ik} [\alpha B]_{kj} && \text{MSM} \\
&= [A(\alpha B)]_{ij} && \text{EMP}
\end{aligned}
$$

So by Definition ME, $(\alpha A)B = A(\alpha B)$.