Math 290 Name: Key Dr. Beezer
Exam 4 Spring 2012
Chapter VS

Show *all* of your work and *explain* your answers fully. There is a total of 100 possible points.
Use Sage only to row-reduce matrices or to solve systems of equations, and be sure to detail your input and output.

1. Determine if the set R below is a linearly independent set in P_2, the vector space of polynomials of degree at most two. (15 points)

$R = \{x^2 + 3x - 5,\ 3x^2 - 8x + 2\}$ RLD: $a_1(x^2+3x-5) + a_2(3x^2-8x+2) = 0x^2+0x+0$

$(a_1+3a_2)x^2 + (3a_1-8a_2)x + (-5a_1+2a_2) = 0x^2+0x+0$

System

$a_1 + 3a_2 = 0$ only solution (Sage) $a_1 = a_2 = 0$

$3a_1 - 8a_2 = 0$ So R is linearly independent

$-5a_1 + 2a_2 = 0$

2. Find a spanning set for the subspace Y of M_{22}. (15 points)

$$Y = \left\{\begin{bmatrix} a & b \\ c & d\end{bmatrix} \in M_{22} \,\middle|\, a = 4c,\ b = -2d\right\} = \left\{\begin{bmatrix} 4c & -2d \\ c & d\end{bmatrix} \,\middle|\, c, d \in \mathbb{C}\right\}$$

$$= \left\{ c\begin{bmatrix} 4 & 0 \\ 1 & 0\end{bmatrix} + d\begin{bmatrix} 0 & -2 \\ 0 & 1\end{bmatrix} \,\middle|\, c, d \in \mathbb{C}\right\} = \left\langle\left\{\begin{bmatrix} 4 & 0 \\ 1 & 0\end{bmatrix}, \begin{bmatrix} 0 & -2 \\ 0 & 1\end{bmatrix}\right\}\right\rangle$$

3. For the matrix A below, compute the rank, nullity and the dimension of the column space. (5 points)

$$A = \begin{bmatrix} 1 & -1 & 6 & 0 & 1 & -1 & -3 \\ 0 & 1 & -3 & 2 & -5 & 5 & -2 \\ -2 & 0 & -6 & -3 & 5 & -5 & 8 \\ 0 & 1 & -3 & 2 & -5 & 5 & -2 \\ 0 & -2 & 6 & -3 & 7 & -7 & 2 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 3 & 0 & 2 & -2 & -1 \\ 0 & 1 & -3 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 1 & -3 & 3 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$r = 3$ rank $= r = 3$

nullity $=$ # columns $-$ rank $= 7 - 3 = 4$

$\dim(\mathcal{C}(A)) =$ rank $= 3$

4. Consider the subspace W of P_3 (the vector space of all polynomials of degree at most 3), and the four elements $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, $\mathbf{y}$ of W. You may assume the following: W is a subspace, all four elements below are in W (except for when you do part (a)), and $B = \{\mathbf{u}, \mathbf{y}\}$ is a basis of W. (35 points)

$$W = \left\{ a + bx + cx^2 + dx^3 \middle| 7a + 5b - 4c + 3d = 0, 4a + 3b - 2c + 2d = 0 \right\}$$

$$\mathbf{u} = 3 + 2x + 4x^2 - 5x^3 \quad \mathbf{v} = -1 + 4x + x^2 - 3x^3 \quad \mathbf{w} = -5 + 2x - 4x^2 + 3x^3 \quad \mathbf{y} = 2 + 6x + 5x^2 - 8x^3$$

(a) Verify that one of $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, $\mathbf{y}$ is an element of W (your choice, just one).

$u \in W$?

$7(3) + 5(2) - 4(4) + 3(-5) = 0$ ✓

$4(3) + 3(2) - 2(4) + 2(-5) = 0$ ✓

(b) Does the set $T = \{\mathbf{w}\}$ span W? Why or why not?

B = basis $\Rightarrow \dim(W) = 2$

$|T| = 1 < 2$ + Goldilocks $\Rightarrow$ T does not span

(c) Is the set $R = \{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ a linearly independent subset of W? Why or why not?

$|R| = 3 > 2 = \dim(W)$ $\Rightarrow$ R linearly dependent.

(d) Does the set $T = \{\mathbf{u}, \mathbf{w}\}$ span W? Why or why not?

T has the "right size". Check linear independence instead.

$\underset{\sim}{0} = a_1 \underset{\sim}{u} + a_2 \underset{\sim}{w}$

$= (3a_1 + (-5)a_2) + (2a_1 + 2a_2)x + (4a_1 - 4a_2)x^2 + (-5a_1 + 3a_2)x^3$

System:

$0 = 3a_1 - 5a_2$

$0 = 2a_1 + 2a_2$

$0 = 4a_1 - 4a_2$

$0 = -5a_1 + 3a_2$

$\Rightarrow$ only solution $a_1 = a_2 = 0$

So T is linearly independent & by Goldilocks theorem, T also spans W.

5. Suppose that V is a vector space. Prove that the zero vector of V is unique. (15 points)

What if there were two zero vectors? Say $\underset{\sim}{0}_1$ & $\underset{\sim}{0}_2$

$\underset{\sim}{0}_1 = \underset{\sim}{0}_1 + \underset{\sim}{0}_2$ because $\underset{\sim}{0}_2$ is a zero vector

$= \underset{\sim}{0}_2$ because $\underset{\sim}{0}_1$ is a zero vector

6. Suppose that A is an $m \times m$ nonsingular matrix and that $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_p\}$ is a linearly independent subset of $\mathbb{C}^m$. Prove that $T = \{A\mathbf{v}_1, A\mathbf{v}_2, \ldots, A\mathbf{v}_p\}$ is a linearly independent subset of $\mathbb{C}^m$. (15 points)

RLD on T: $a_1 A\underset{\sim}{v}_1 + a_2 A\underset{\sim}{v}_2 + \cdots + a_p A\underset{\sim}{v}_p = \underset{\sim}{0}$

$A(a_1\underset{\sim}{v}_1 + a_2\underset{\sim}{v}_2 + \cdots + a_p\underset{\sim}{v}_p) = \underset{\sim}{0}$ ← several steps w/ obvious theorems.

A nonsingular $\Rightarrow$ $a_1\underset{\sim}{v}_1 + a_2\underset{\sim}{v}_2 + \cdots + a_p\underset{\sim}{v}_p = \underset{\sim}{0}$

Because S is linearly independent, we can conclude that $a_1 = a_2 = \cdots = a_p = 0$.

Therefore T is linearly independent.