Math 290 Name: Key Dr. Beezer
Exam 5 Spring 2012
Chapters D, E

Show *all* of your work and *explain* your answers fully. There is a total of 100 possible points.
Pay attention to specific instructions about the use of Sage on relevant problems.

1. Find the determinant of the matrix A using expansion about a row or column, and doing all the computations by hand. (15 points)

$$A = \begin{bmatrix} 4 & 3 & -1 \\ -4 & 4 & 5 \\ -3 & 3 & 1 \end{bmatrix}$$

$$\det(A) = 4\begin{vmatrix} 4 & 5 \\ 3 & 1 \end{vmatrix} + (-1)(3)\begin{vmatrix} -4 & 5 \\ -3 & 1 \end{vmatrix} + (-1)\begin{vmatrix} -4 & 4 \\ -3 & 3 \end{vmatrix}$$

$$= 4(-11) + (-3)(-11) + (-1)(0) = 7(-11) = -77$$

expansion about row 1

2. Consider the matrix B. (20 points)

$$B = \begin{bmatrix} 10 & -5 & 5 & 13 & -14 \\ 24 & -29 & 20 & 56 & -46 \\ 27 & -39 & 28 & 73 & -60 \\ 19 & -21 & 15 & 44 & -37 \\ 24 & -27 & 20 & 56 & -48 \end{bmatrix}$$

(a) Use Sage to compute the characteristic polynomial of B and then use Sage to analyze the characteristic polynomial and obtain the eigenvalues of B along with their algebraic multiplicities.

$p = B.\text{charpoly}() = x^5 - 5x^4 - 5x^3 + 45x^2 - 108$

$p.\text{factor}() = (x+2)^2(x-3)^3 = (x-(-2))^2(x-3)^3$

Eigenvalues: $-2, 3$ $\quad \gamma_B(-2) = 2 \quad \alpha_B(3) = 3$

(b) Compute bases for the eigenspaces of B, and the geometric multiplicity of each eigenvalue, using Sage **only** to row-reduce matrices (and form simple linear combinations of matrices).

$$(B - (-2)I_5).\text{rref}() \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & -1/4 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -5/4 \\ 0 & 0 & 0 & 1 & -3/4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad E_B(-2) = \left\langle\left\{\begin{bmatrix} 1/4 \\ 1 \\ 5/4 \\ 3/4 \\ 1 \end{bmatrix}\right\}\right\rangle$$

$\gamma_B(-2) = 1$

$$(B - 3I_5).\text{rref}() \rightarrow \begin{bmatrix} 1 & 0 & 0 & -1 & 1/2 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 4 & -9/2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad E_B(3) = \left\langle\left\{\begin{bmatrix} 1 \\ 0 \\ -4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/2 \\ 1 \\ 9/2 \\ 0 \\ 1 \end{bmatrix}\right\}\right\rangle$$

$\gamma_B(3) = 2$

3. Diagonalize the matrix C below by giving the matrix for the similarity transformation and the resulting diagonal matrix. Use **only** the Sage `.eigenspaces_right()` method and show clearly how you are using the output. (20 points)

$$C = \begin{bmatrix} -16 & 21 & 42 & 42 \\ -14 & 19 & 28 & 28 \\ 7 & -7 & -9 & -14 \\ -7 & 7 & 14 & 19 \end{bmatrix}$$

$$D = \begin{bmatrix} -2 & & & 0 \\ & 5 & & \\ & & 5 & \\ 0 & & & 5 \end{bmatrix} \qquad S = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 2/3 & 0 & 1 & 0 \\ -1/3 & 0 & 0 & 1 \\ 1/3 & 1/2 & -1/2 & -1 \end{bmatrix}$$

eigenvalues from Sage output

basis vector for eigenspace for $\lambda = -2$

basis vectors for eigenspace for $\lambda = 5$

4. Suppose that $\mathbf{x}$ and $\mathbf{y}$ are eigenvectors for the same eigenvalue λ of the matrix A. Prove directly that $\mathbf{x} + \mathbf{y}$ is also an eigenvector for the eigenvalue λ of the matrix A. Here, "directly" means you are proving a closure property, and to not use the fact that eigenspaces are vector spaces. (15 points)

Known: $A\underset{\sim}{x} = \lambda \underset{\sim}{x}$ & $A\underset{\sim}{y} = \lambda \underset{\sim}{y}$

Then: $A(\underset{\sim}{x} + \underset{\sim}{y}) = A\underset{\sim}{x} + A\underset{\sim}{y}$ MMDAA

$= \lambda \underset{\sim}{x} + \lambda \underset{\sim}{y}$ Hypothesis

$= \lambda(\underset{\sim}{x} + \underset{\sim}{y})$ Property SMDAA

This says that $\underset{\sim}{x} + \underset{\sim}{y}$ is an eigenvector of A for λ.

(But what if $\underset{\sim}{y} = -\underset{\sim}{x}$??)

5. Suppose that A has $\lambda = 4$ as an eigenvalue. Define the polynomial $p(x) = 2x^3 - 6x - 31$. Find, with proof, an eigenvalue of $p(A)$. (15 points)

Let $\underset{\sim}{x}$ be an eigenvector of A for $\lambda=4$.

Then
$$
\begin{aligned}
p(A)\underset{\sim}{x} &= (2A^3 - 6A - 31I)\underset{\sim}{x} \\
&= (2\lambda^3 - 6\lambda - 31)\underset{\sim}{x} \\
&= (2(4)^3 - 6(4) - 31)\underset{\sim}{x} \\
&= 73\underset{\sim}{x}
\end{aligned}
$$
Theorem EPM

So $\rho = 73$ is an eigenvalue of $p(A)$.

6. Use Sage to create some small, but nontrivial, square matrices over the rationals. For each matrix, compute the characteristic polynomial, and then evaluate the polynomial with the matrix. Document your experiments by giving the relevant commands employed and the output created. What do you observe? (15 points)

```
A = random_matrix(QQ, 20)
p = A.charpoly()
p(A)
```

These commands will output a zero matrix every time. This is the Cayley-Hamilton Theorem.