Math 290 Name: Key Dr. Beezer
Exam 6 Spring 2012
Chapter LT

Show *all* of your work and *explain* your answers fully. There is a total of 100 possible points.
Be certain that all computations can be justified by definitions and theorems we have covered. You may use Sage to row-reduce matrices and solve systems of equations.

1. Verify that the function below is a linear transformation. P_1 is the vector space of polynomials with degree at most 1 and M_{12} is the vector space of 1×2 matrices. (15 points)

   $T\colon P_1 \to M_{12}, \quad T(a + bx) = \begin{bmatrix} 2a + b & a - 4b \end{bmatrix}$

$$\begin{aligned} T((a+bx)+(c+dx)) &= T((a+c)+(b+d)x) \\ &= [2(a+c)+(b+d) \quad a+c-4(b+d)] \\ &= [(2a+b)+(2c+d) \quad (a-4b)+(c-4d)] \\ &= [2a+b \quad a-4b] + [2c+d \quad c-4d] \\ &= T(a+bx) + T(c+dx) \end{aligned}$$

$$T(\alpha(a+bx)) = T(\alpha a + \alpha bx) = [2(\alpha a)+\alpha b \quad \alpha a - 4(\alpha b)]$$
$$= [\alpha(2a+b) \quad \alpha(a-4b)] = \alpha[2a+b \quad a-4b] = \alpha T(a+bx)$$

2. The linear transformation S is invertible (you may assume this). Compute three pre-images for S, one for each of the standard unit vectors of $\mathbb{C}^3$. Use these pre-images to construct the inverse linear transformation, S^{-1}. (20 points)

   $S\colon \mathbb{C}^3 \to \mathbb{C}^3, \quad S\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} a - 3b + 5c \\ -a + 4b - 6c \\ -2a + 2b - 5c \end{bmatrix}$

$$S^{-1}\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)? \Rightarrow S\left(\begin{bmatrix}a\\b\\c\end{bmatrix}\right) = \begin{bmatrix}1\\0\\0\end{bmatrix} \Rightarrow \begin{array}{l} a-3b+5c=1 \\ -a+4b-6c=0 \\ -2a+2b-5c=0 \end{array} \Rightarrow \begin{array}{l} a=-8 \\ b=7 \\ c=6 \end{array} \Rightarrow S^{-1}\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}-8\\7\\6\end{bmatrix}$$

Similarly,

$$S^{-1}\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}-5\\5\\4\end{bmatrix}, \quad S^{-1}\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}-2\\1\\1\end{bmatrix}, \text{ so}$$

$$S^{-1}\left(\begin{bmatrix}f\\g\\h\end{bmatrix}\right) = S^{-1}\left(f\begin{bmatrix}1\\0\\0\end{bmatrix} + g\begin{bmatrix}0\\1\\0\end{bmatrix} + h\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = fS^{-1}\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) + gS^{-1}\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) + hS^{-1}\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)$$

$$= f\begin{bmatrix}-8\\7\\6\end{bmatrix} + g\begin{bmatrix}-5\\5\\4\end{bmatrix} + h\begin{bmatrix}-2\\1\\1\end{bmatrix} = \begin{bmatrix}-8f-5g-2h\\7f+5g+h\\6f+4g+h\end{bmatrix}$$

3. Consider the linear transformation R whose domain is M_{22}, the vector space of 2×2 matrices and whose codomain is P_2, the vector space of polynomials with degree at most 2. (35 points)

$$R\colon M_{22} \to P_2, \quad R\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a + 2b + c - 4d) + (-a - b + 2c + d)\,x + (-3a - 4b + 4c + 5d)\,x^2$$

(a) Compute the kernel of R, $\mathcal{K}(R)$.

$R\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = 0 + 0x + 0x^2$??

$$\Rightarrow \begin{aligned} a + 2b + c - 4d &= 0 \\ -a - b + 2c + d &= 0 \\ -3a - 4b + 4c + 5d &= 0 \end{aligned}$$

rref of coefficient matrix

$$\begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$\Rightarrow a = 3d,\ b = 0,\ c = d$

So $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 3d & 0 \\ d & d \end{bmatrix} = d\begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix}$

$\text{Ker } R = \left\langle \left\{ \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix} \right\} \right\rangle$

(b) Compute the range of R, $\mathcal{R}(R)$

Given $f + gx + hx^2$ is there $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $R\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = f + gx + hx^2$?

Is the following system consistent?

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & -4 & f \\ -1 & -1 & 2 & 1 & g \\ -3 & -4 & 4 & 5 & h \end{array}\right]$$

Coefficient matrix row-reduces as above, so column space is all of $\mathbb{C}^3$

System is always consistent, so $\mathcal{R}(R) = P_2$, ie for any $f + gx + hx^2$ the pre-image is non-empty.

(c) Is R injective? Why or why not?

No, $\mathcal{K}(R) \neq \{0\}$.

(d) Is R surjective? Why or why not?

Yes, $\mathcal{R}(R) = P_2$

(e) If R is not injective, find two different nonzero vectors, $\mathbf{x}$ and $\mathbf{y}$, such that $R(\mathbf{x}) = R(\mathbf{y})$.

$x = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ (any non zero matrix)

$y = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix}$ (x + kernel-element)

$\Rightarrow R(x) = R(y) = 1 - x - 3x^2$

(f) If R is not surjective, find a vector $\mathbf{w}$ in the codomain of R that is not in the range of R.

N/A

Note: This check is not necessary if you build $\underset{\sim}{x}$ & $\underset{\sim}{y}$ correctly

4. Suppose U is a vector space and $\rho \in \mathbb{C}$ is a scalar. Define a function $T_\rho \colon U \to U$ by $T_\rho(\mathbf{u}) = \rho\mathbf{u}$. Prove that T_ρ is a linear transformation. Be sure to provide justification/explanation for each step of your proof. (15 points)

$$\begin{aligned}
T_\rho(\underset{\sim}{x} + \underset{\sim}{y}) &= \rho(\underset{\sim}{x} + \underset{\sim}{y}) && \text{Defn } T_\rho \\
&= \rho \underset{\sim}{x} + \rho \underset{\sim}{y} && \text{Property DVA} \\
&= T_\rho(\underset{\sim}{x}) + T_\rho(\underset{\sim}{y}) && \text{Defn } T_\rho
\end{aligned}$$

$$\begin{aligned}
T_\rho(\alpha \underset{\sim}{x}) &= \rho(\alpha \underset{\sim}{x}) && \text{Defn } T_\rho \\
&= (\rho\alpha)\underset{\sim}{x} && \text{Property SMA} \\
&= (\alpha\rho)\underset{\sim}{x} && \text{Property CMCN} \\
&= \alpha(\rho \underset{\sim}{x}) && \text{Property SMA} \\
&= \alpha T_\rho(\underset{\sim}{x}) && \text{Defn } T_\rho
\end{aligned}$$

5. Suppose that $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n\}$ is a basis of the vector space U, and that S and T are linear transformations that both have U as their domain. Suppose further that S and T agree on the basis – that is, $S(\mathbf{u}_i) = T(\mathbf{u}_i)$ for $1 \leq i \leq n$. Prove that S and T are the same function. (15 points)

Let $\underset{\sim}{u} \in U$ be an arbitrary element of U.

Then there are scalars $a_1, a_2, \ldots, a_n$;

$\underset{\sim}{u} = a_1 \underset{\sim}{u_1} + a_2 \underset{\sim}{u_2} + \cdots + a_n \underset{\sim}{u_n}$ so

$$\begin{aligned}
S(\underset{\sim}{u}) &= S(a_1 \underset{\sim}{u_1} + \cdots + a_n \underset{\sim}{u_n}) \\
&= a_1 S(\underset{\sim}{u_1}) + \cdots + a_n S(\underset{\sim}{u_n}) && \text{LTLC} \\
&= a_1 T(\underset{\sim}{u_1}) + \cdots + a_n T(\underset{\sim}{u_n}) && \text{Hypothesis} \\
&= T(a_1 \underset{\sim}{u_1} + \cdots + a_n \underset{\sim}{u_n}) && \text{LTLC} \\
&= T(\underset{\sim}{u})
\end{aligned}$$

So S & T have the same value for each possible input, hence $S = T$.