Math 290 | Name: Key | Dr. Beezer
Exam 7 | | Spring 2012
Chapter R

Show *all* of your work and *explain* your answers fully. There is a total of 100 possible points.
You may use Sage to row-reduce matrices, solve systems of equations, compute determinants and compute eigen-stuff. Linear transformation routines may not be used as justification.

1. Compute a matrix representation of the linear transformation T with domain P_1, the vector space of polynomials with degree at most 1, and codomain M_{13}, the vector space of 1×3 matrices. Then illustrate the Fundamental Theorem of Matrix Representation (FTMR) by using the representation to compute $T(3-6x)$. (No credit will be given for using other methods to compute this output of the linear transformation.) (15 points)

   $T\colon P_1 \to M_{13},\quad T(a+bx) = \begin{bmatrix} 2a+b & a-4b & 5a+6b \end{bmatrix}$

Lets choose "nice" bases. $B = \{1, x\}$, $C = \{[1\,0\,0], [0\,1\,0], [0\,0\,1]\}$

$$\rho_C(T(1)) = \rho_C([2\ 1\ 5]) = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}$$
$$\rho_C(T(x)) = \rho_C([1\ {-4}\ 6]) = \begin{bmatrix} 1 \\ -4 \\ 6 \end{bmatrix} \Rightarrow M^T_{B,C} = \begin{bmatrix} 2 & 1 \\ 1 & -4 \\ 5 & 6 \end{bmatrix}$$

$$T(3-6x) = \rho_C^{-1}\left(M^T_{B,C}\,\rho_B(3-6x)\right) = \rho_C^{-1}\left(\begin{bmatrix} 2 & 1 \\ 1 & -4 \\ 5 & 6 \end{bmatrix}\begin{bmatrix} 3 \\ -6 \end{bmatrix}\right)$$
$$= \rho_C^{-1}\left(\begin{bmatrix} 0 \\ 27 \\ -21 \end{bmatrix}\right) = 0[1\,0\,0] + 27[0\,1\,0] + (-21)[0\,0\,1]$$
$$= [0\ \ 27\ \ {-21}]$$

2. For the linear transformation below, find a basis of the vector space P_2 so that the matrix representation of S relative to the basis is a diagonal matrix. Give the ensuing representation as well. (20 points)

   $S\colon P_2 \to P_2,\quad S(a+bx+cx^2) = (-12a-4b+14c) + (9a+6b-9c)\,x + (-9a-4b+11c)\,x^2$

the obvious representation is $M = \begin{bmatrix} -12 & -4 & 14 \\ 9 & 6 & -9 \\ -9 & -4 & 11 \end{bmatrix}$

Eigenvalues & eigenvectors:

$\lambda = 6 \quad \mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

$\lambda = 2 \quad \mathbf{x}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$\lambda = -3 \quad \mathbf{x}_3 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

Basis is uncoordinatized eigenvectors:

$B = \{1 - x + x^2,\ 1 + x^2,\ 2 - x + x^2\}$

Representation has eigenvalues on diagonal:

$$M^S_{B,B} = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

3. Consider the linear transformation R with domain M_{12}, the vector space of 1×2 matrices, and codomain P_1, the vector space of polynomials with degree at most 1. (35 points)

$R\colon M_{12} \to P_1, \quad R\left(\begin{bmatrix} a & b \end{bmatrix}\right) = (3a + 7b) + (2a + 5b)\,x$

(a) Build a matrix representation of R relative to the bases $B = \left\{\begin{bmatrix} 3 & 2 \end{bmatrix}, \begin{bmatrix} 4 & 3 \end{bmatrix}\right\}$ and $C = \{6 + 5x,\, 1 + x\}$.

$$\rho\left(R([3\ 2])\right) = \rho_C(23+16x) = \rho_C\left(7(6+5x) + (-19)(1+x)\right) = \begin{bmatrix} 7 \\ -19 \end{bmatrix}$$

$$\rho\left(R([4\ 3])\right) = \rho_C(33+23x) = \rho_C\left(10(6+5x) + (-27)(1+x)\right) = \begin{bmatrix} 10 \\ -27 \end{bmatrix}$$

$$M^R_{B,C} = \begin{bmatrix} 7 & 10 \\ -19 & -27 \end{bmatrix}$$

(b) R is invertible (you may assume this). Compute the matrix representation of R^{-1} relative the bases C and B given in part (a).

$$M^{R^{-1}}_{C,B} = \left(M^R_{B,C}\right)^{-1} = \begin{bmatrix} 7 & 10 \\ -19 & -27 \end{bmatrix}^{-1} = \begin{bmatrix} -27 & -10 \\ 19 & 7 \end{bmatrix}$$

(c) Consider two new bases, $X = \left\{\begin{bmatrix} 7 & 3 \end{bmatrix}, \begin{bmatrix} 2 & 1 \end{bmatrix}\right\}$ and $Y = \{1 + 2x,\, -1 - x\}$. Form the change-of-basis matrix, $C_{B,X}$, from basis B to basis X.

$$\rho_X([3\ 2]) = \rho_X\left((-1)[7\ 3] + 5[2\ 1]\right) = \begin{bmatrix} -1 \\ 5 \end{bmatrix}$$

$$\rho_X([4\ 3]) = \rho_X\left((-2)[7\ 3] + 9[2\ 1]\right) = \begin{bmatrix} -2 \\ 9 \end{bmatrix}$$

so $$C_{B,X} = \begin{bmatrix} -1 & -2 \\ 5 & 9 \end{bmatrix}$$

(d) Compute the matrix representation of R relative to the bases X and Y (in part (c)) by using the representation from part (a) and change-of-basis matrices. No credit will be given for building this representation via the definition of a matrix representation.

Change-of-basis, $C \to Y$

$$\rho_Y(6+5x) = \rho_Y\left((-1)(1+2x) + (-7)(-1-x)\right) = \begin{bmatrix} -1 \\ -7 \end{bmatrix} \qquad C_{C,Y} = \begin{bmatrix} -1 & 0 \\ -7 & -1 \end{bmatrix}$$

$$\rho_Y(1+x) = \rho_Y\left(0(1+2x) + (-1)(-1-x)\right) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$M^R_{X,Y} = C_{C,Y}\, M^R_{B,C}\, C_{X,B} = C_{C,Y}\, M^R_{B,C}\, C_{B,X}^{-1}$$

$$= \begin{bmatrix} -1 & 0 \\ -7 & -1 \end{bmatrix} \begin{bmatrix} 7 & 10 \\ -19 & -27 \end{bmatrix} \begin{bmatrix} -1 & -2 \\ 5 & 9 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} -13 & -4 \\ -55 & -17 \end{bmatrix}$$

4. We saw the following linear transformation on the previous exam. Suppose U is a vector space and $\rho \in \mathbb{C}$ is a scalar. Define the linear transformation $T_\rho\colon U \to U$ by $T_\rho(\mathbf{u}) = \rho\mathbf{u}$. Describe, with justification, a matrix representation of T. (15 points)

$B = \{\underset{\sim}{u_1}, \underset{\sim}{u_2}, \ldots, \underset{\sim}{u_n}\}$ some basis of U.

$$\rho_B(T_\rho(\underset{\sim}{u_i})) = \rho_B(\rho \underset{\sim}{u_i}) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \rho \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow \rho \text{ in slot } i$$

So

$$M^{T_\rho}_{B,B} = \begin{bmatrix} \rho & & & 0 \\ & \rho & & \\ & & \rho & \\ 0 & & & \ddots \\ & & & & \rho \end{bmatrix}$$

5. Suppose that U is a vector space, B is a basis of U, $T\colon U \to U$ is a linear transformation, and $\mathbf{u} \in U$ is an eigenvector of T. Prove that the vector representation $\rho_B(\mathbf{u})$ is an eigenvector for the matrix representation $M^T_{B,B}$. (15 points)

$$M^T_{B,B}\,\rho_B(\underset{\sim}{u}) = \rho_B(T(\underset{\sim}{u})) \qquad \text{FTMR (C in B)}$$

$$= \rho_B(\lambda \underset{\sim}{u}) \qquad \underset{\sim}{u} \text{ eigenvector}$$

$$= \lambda\,\rho_B(\underset{\sim}{u}) \qquad \rho_B \text{ linear transformation}$$