Math 290 Name: Dr. Beezer
Exam 6 Spring 2012
Chapter LT

Show *all* of your work and *explain* your answers fully. There is a total of 100 possible points.
Be certain that all computations can be justified by definitions and theorems we have covered. You may use Sage to row-reduce matrices and solve systems of equations.

1. Verify that the function below is a linear transformation. P_1 is the vector space of polynomials with degree at most 1 and M_{12} is the vector space of 1×2 matrices. (15 points)

   $$T\colon P_1 \to M_{12}, \quad T(a+bx) = \begin{bmatrix} 2a+b & a-4b \end{bmatrix}$$

2. The linear transformation S is invertible (you may assume this). Compute three pre-images for S, one for each of the standard unit vectors of $\mathbb{C}^3$. Use these pre-images to construct the inverse linear transformation, S^{-1}. (20 points)

   $$S\colon \mathbb{C}^3 \to \mathbb{C}^3, \quad S\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} a-3b+5c \\ -a+4b-6c \\ -2a+2b-5c \end{bmatrix}$$

3. Consider the linear transformation R whose domain is M_{22}, the vector space of 2×2 matrices and whose codomain is P_2, the vector space of polynomials with degree at most 2. (35 points)

$$R\colon M_{22} \to P_2, \quad R\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a + 2b + c - 4d) + (-a - b + 2c + d)\,x + (-3a - 4b + 4c + 5d)\,x^2$$

(a) Compute the kernel of R, $\mathcal{K}(R)$.

(b) Compute the range of R, $\mathcal{R}(R)$

(c) Is R injective? Why or why not?

(d) Is R surjective? Why or why not?

(e) If R is not injective, find two different nonzero vectors, $\mathbf{x}$ and $\mathbf{y}$, such that $R(\mathbf{x}) = R(\mathbf{y})$.

(f) If R is not surjective, find a vector $\mathbf{w}$ in the codomain of R that is not in the range of R.

4. Suppose U is a vector space and $\rho \in \mathbb{C}$ is a scalar. Define a function $T_\rho \colon U \to U$ by $T_\rho(\mathbf{u}) = \rho\mathbf{u}$. Prove that T_ρ is a linear transformation. Be sure to provide justification/explanation for each step of your proof. (15 points)

5. Suppose that $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n\}$ is a basis of the vector space U, and that S and T are linear transformations that both have U as their domain. Suppose further that S and T agree on the basis – that is, $S(\mathbf{u}_i) = T(\mathbf{u}_i)$ for $1 \leq i \leq n$. Prove that S and T are the same function. (15 points)